AN ANALYSIS OF SIX MAJOR ARTICLES ON TONEARM ALIGNMENT OPTIMISATION AND A SUMMARY OF OPTIMUM DESIGN EQUATIONS

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The latest revision of this book may be downloaded from Vinyl Engine at:
https://www.vinylengine.com/turntable_forum/viewtopic.php?f=46&t=4854

TABLE OF CONTENTS

INTRODUCTION

The purpose of this book on tonearm alignment optimisation is three-fold:

1. To consider in detail the design formulas presented in the six major articles on tonearm alignment optimisation.
2. To compare the design formulas.
3. To provide a comprehensive listing of design formulas.

The book commences by taking the design formulas presented in the six articles, then reformulating them to a common format and notation. A comparison is then made between the formulas in the six articles, and finally, a comprehensive listing of design formulas is provided.

Although I believe this book provides information in a form which has not appeared before, I make no claim to providing new material. My thanks to David Mohr for his helpful comments.

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March 1983

SIX MAJOR ARTICLES ANALYSED

The following six major articles on optimum tonearm alignment geometry have been analysed and compared:

Löfgren, E.G. [1]
"Über die nichtlineare Verzerrung bei der Wiedergabe von Schallplatten infolge Winkelabweichungen des Abtastorgans", *Akustische Zeitschrift*, Nov. 1938, pg. 350.

A translation of the German article was undertaken by Klaus Rampelmann in 2008:

"On The Non-Linear Distortion In The Reproduction Of Phonograph Records Caused By Angular Deviation Of The Pick-up Needle".

It may be downloaded at: http://www.vinylengine.com/phpBB2/viewtopic.php?t=15875

Baerwald, H.G. [2]
"Analytic Treatment Of Tracking Error And Notes On Optimal Pickup Design", *Journal of Society of Motion Picture Engineers*, Dec. 1941, pg. 591. Corrections are presented on page S3-12 of this book.

Bauer, B.B. [3]
"Tracking Angle In Phonograph Pickups", *Electronics*, March 1945, pg. 110.

Seagrave, J.D. [4]
"Minimising Pickup Tracking Error", *Audiocraft Magazine*, Dec. 1956, pg. 19 (Part 1); Jan. 1957, pg. 25 (Part 2); Aug. 1957, pg. 22 (A Sequel). Corrections are presented on page S5-9 of this book.

Stevenson, J.K. [5]
"Pickup Arm Design", *Wireless World*, May 1966, pg. 214 (Part 1); June 1966, pg. 314 (Part 2); Corrections: Aug. 1966, pg. 396.

Kessler, M.D. and Pisha, B.V. [6]
"Tonearm Geometry And Setup Demystified", *Audio*, Jan. 1980, pg. 76; Errata: April 1980, pg. 26.

HISTORY BEHIND THIS BOOK

Löfgren's Article: In March 1983, the writer obtained a copy of Professor Erik Löfgren's 1938 German-language paper [1]. It was the trigger behind the writer's letter to the editor of *Audio* which was published in the May 1983 edition [7].

The first release of this book was in March 1983. It contained 92 pages. Further releases have since been made. This release contains 263 pages and much of Section 1 has been revised.

Prior to the September 2015 release of this book, the earliest reference applying offset angle and overhang principles known to the writer was the two-part article by Percy Wilson in the September and October 1924 issues of *The Gramophone* [8]. Wilson shows how to employ offset angle and overhang in the setting up of a tonearm for the purposes of maximising sound quality and minimising record wear by minimising tracking error. More follows.

Recent Facts: Klaus Rampelmann located two earlier French patents which may be the earliest attempts at addressing tracking distortion and record wear. These are French patents 385526 (1908) to Bela Harsanyi (Hungary) [9] and 422638 (1911) to Louis Lumière (France) [10], both of which minimise tracking error. Harsanyi developed equations for the application of offset angle and overhang for two null points, and Lumière developed a geometrical solution for two null points without providing formulas. Harsanyi's patent precedes Wilson's article by sixteen years.

Tracking Error Minimisation: The section in this book titled Alignment for Tracking Error Minimisation has been completely rewritten and expanded to include the details behind the alignment developments which minimised *tracking error*, i.e., alignment articles published *before* Löfgren's ground-breaking article in 1938 which minimised *weighted* tracking error. The works of Harsanyi, Lumière, Wilson and a 1929 Swedish article by Löfgren [11], recently located and translated by Klaus Rampelmann, have been analysed in detail. The overhang formulas of Harsanyi, Wilson and Löfgren (1929) are identical, while their offset angle formulas are slightly different. More follows.

'Löfgren A' Alignment: This is Löfgren's **prime** alignment strategy which is the historical and fundamental focus of his 1938 article. *This solution changed tonearm alignment forever!* It is based on adjus*ting the offset angle and overhang to minimise and equalise the magnitudes of the three peaks of the weighted tracking error (WTE) to minimise the maximum tracking distortion.* Löfgren provides solutions for the required offset angle and overhang. More follows.

'Löfgren B' Alignment: This alignment strategy is Löfgren's **compromise** alignment, used **only** when the offset angle is fixed and non-optimum, which would have been the case with most, if not all, of the gramophones on the planet at the time! This alignment offers a solution based on *adjusting the overhang to achieve minimum LMS/RMS tracking distortion*. Löfgren provides a solution for the distortion-minimising overhang. This was Löfgren's compromise alignment for dealing with this less-than-ideal situation. (Note: If the offset angle is adjustable, the 'Löfgren A' alignment is used.) More follows.

The 'Löfgren B' solution would have been beneficial to manufacturers, retailers, repairers and owners of gramophones fitted with tonearms whose offset angle had been set in manufacture at an angle other than optimum and where some adjustment of the arm mounting position was possible as a way of adjusting the overhang. After applying the 'Löfgren B' overhang, those gramophones would have produced improved sound quality, which would have been of huge interest to music lovers with a passion for high fidelity sound reproduction from their gramophones.

'Löfgren C' Alignment: This alignment strategy is Löfgren's **alternative** alignment to the 'Löfgren A' alignment. Löfgren considered that objections could be raised against the relatively slowly changing WTE (and distortion) around the central WTE peak of the 'Löfgren A' alignment. This concern was addressed by this alternative alignment strategy which aims at lowering the magnitude of the central WTE peak, although increasing the WTE at the inner and outer radii. It is based on *adjusting the offset angle and overhang to achieve minimum LMS/RMS tracking distortion.* Löfgren does not provide a solution for this alternative strategy, but we can calculate these using tools such as the Microsoft Excel *Solver*. More follows.

English Translation of Löfgren's 1938 Article
After many decades since its publishing, the historic 1938 article by Löfgren was translated into English in 2008 by Klaus Rampelmann. I wish to express my gratitude to Klaus for taking on this complex, highly detailed linguistic task and for making the translation freely accessible via the internet. At long last, Löfgren's article may be read in full detail by those who are language-challenged like this writer. The link to the translation is provided on the References page.

Corrections to Alignment Strategies and Terminology: New insights into Löfgren's 1938 article were gained by this writer in 2015. This resulted in **two** corrections being made in the 2015 release of this book. The first correction resulted after the writer identified a third alignment strategy from Löfgren, the **'Löfgren C'** alignment, which was added to this book.

The second correction resulted after the writer correctly understood the true purpose of the **'Löfgren B'** alignment. The writer's original understanding was to set the offset angle equal to the 'Löfgren A' offset angle (incorrect), then apply Löfgren's EQNs (47) and (48) to calculate the required overhang to minimise the LMS/RMS tracking distortion (correct), for the purpose of reducing the magnitude of the central WTE peak (incorrect). Löfgren's purpose for EQNs (47) and (48) was to apply them in circumstances where the tonearm has a *fixed, non-optimum* offset angle to provide a **compromise** alignment which minimises the LMS/RMS distortion in this non-ideal situation. The 'Löfgren B' strategy is further successful for this purpose because tracking distortion is more sensitive to errors in overhang than to errors in offset angle. More follows. (If the offset angle is adjustable, the 'Löfgren A' alignment is used as already noted.)

The writer's failure to identify these two issues began with the first release of this book in 1983. The writer deeply regrets and sincerely apologises for failing to identify these errors earlier, as it would have prevented the errors in terminology and application of Löfgren's alignments which have occurred, and continue to occur, in some internet forums and with some alignment tools which apply Löfgren's alignment strategies. The writer hopes the clarifications here will provide the impetus for such errors to be corrected.

Final notes: The comprehensive listing of equations in Section S9 forms a practical compendium for those designing and mounting tonearms and cartridges. It is useful when confirming manufacturers' alignment settings or for seeking further understanding about the subject.

Sections S10, S11 and S12 provide the detailed derivation of many of the equations listed in Section S9, including Löfgren's equations underpinning the 'Löfgren A' and 'Löfgren B' alignments. Löfgren's LMS/RMS distortion-minimising equation, which forms the basis of the 'Löfgren B' alignment, is also derived.

Included are the spreadsheet settings for calculating the "perfect" (high accuracy) 'Löfgren A', 'Löfgren B', 'Löfgren C' and 'Stevenson A' alignment settings. The spreadsheet calculations used for calculating LMS/RMS distortion using the Composite Trapezoidal Rule are also described.

After the writer obtained a copy of Löfgren's 1938 article, several key concerns and questions arose regarding Baerwald and his 1941 article. These are discussed at the end of Section S1.

Finally, to quote a fundamental reminder from Keith Howard [12]: "*The record groove is a spiral, and therefore its tangent is never truly at a right angle to a radial line drawn from the centre of the disc through the stylus. Whilst true, this is irrelevant to the alignment issue, since the aim is not to align the cartridge front-back axis as closely as possible to a true tangent to the groove, but to align it as closely as possible to the front-back axis of the cutting head, which is aligned (nominally, at least) at a right angle to the disc radius*."

Please note: Throughout this book, the terms *tracking error* and *tracking distortion* are used as a shorthand for *lateral tracking error* and *lateral tracking error distortion,* respectively.

Advice of errors or suggestions for improvements are most welcome and may be sent to the writer at gdennes@gmail.com.

This book is accompanied by an Excel tonearm alignment spreadsheet written by the writer.
Graeme F. Dennes
14 May 2024

HISTORY AND THE FACTS

The Harsanyi Patent: Maximising Sound Quality and Minimising Record Wear
Over the last century, several important contributions on the subject of tonearm alignment geometry have been published. The earliest use of offset angle and overhang principles known to the writer is presented in a 1908 French patent [9]. The patent number is 385526 in the name of Bela Harsanyi of Hungary and was published on 15 May 1908. The patent describes the application of offset angle and overhang principles for purposes of *maximising sound quality, minimising background noise and minimising record wear*. Formulas for the required offset angle and overhang are provided.

Sound quality
Tracking error in pivoted arms occurs whenever the vertical plane (the stylus plane) through the stylus pivot point and the at-rest stylus tip is not at right-angles to the radial line drawn from the centre of the disc through the stylus tip. The left-right motion of the stylus tip should be along a radial line to the disc centre during play, as occurs with the original cutting stylus. This is achieved, by definition, with a linear-tracking/parallel-tracking arm, but not with the pivoted arm. When this requirement is not fulfilled, distortion of the reproduced sound occurs.

Record Wear
Very few phonograph needles at the time possessed a point fine enough to reach the bottom of the record grooves, so the needle would usually ride up on the groove walls. As the (round) needle tip follows the groove during play, the wear from the groove walls produces facets on the sides of the needle, resulting in the creation of a chisel-shaped tip along the line of the groove. As the arm traverses the record during play, the tracking angle is slowly but *continually* changing due to the geometrical arrangements. The chisel-like shape of the needle, which is continually being reground due to the continuing change in the tracking angle across the playing surface, produces extra wear on the record, due primarily to the leading edge of the needle's chisel tip being continually driven into the groove walls, causing the facets to be reground to "fit" the groove walls. This process is ongoing and relentless during play and the groove walls become damaged and worn from the chisel-like shape of the needle tip and the surface noise increases.

The Doubly-Good Solution
To maximise the sound quality and minimise the record wear, the method devised by Harsanyi was firstly to apply an overhang which minimises the *change* in tracking angle so as to *minimise* the angular "rotation" of the needle in the groove across the playing surface, which in turn *minimises the reshaping of the needle tip* and the subsequent damage to the groove walls from the chisel front edge of the needle, which in turn *minimises record wear*. Then secondly, to apply an offset angle which *minimises* the tracking error and so *maximises sound quality* by minimising distortion.

The Wilson Article: Maximising Sound Quality and Minimising Record Wear
The work of Percy Wilson [8] in 1924 also describes the application of offset angle and overhang principles in a manner similar to Harsanyi to maximise sound quality and minimise record wear through minimising tracking error. Wilson's overhang formula is identical to Harsanyi's, while Wilson's offset angle formula results in half the tracking error of Harsanyi's, and so provides for better sound quality. Wilson's article is the earliest article known to the writer which presents an analysis of tracking error and its minimisation. Wilson also provides extensive practical advice on the subject of tonearms and their setup.

In 1925, Wilson published the design of an alignment protractor [13] based on [8].

The first statement known to the writer alluding to the origin of tracking distortion was made in 1937 by Bird and Chorpening [14]. In the presence of tracking error, "*... since the needle point must vibrate about an axis at an angle to the groove tangent, sinusoidal modulation of the groove will not produce sinusoidal vibration of the needle point even when close contact is maintained, and therefore a certain amount of waveform distortion occurs with a large tracking error*".

This was followed later in 1937 by Olney [15], who developed a model for tracking distortion, and postulated that tracking distortion would be related to the ratio of the recorded *amplitude* of the groove modulation to the recorded *wavelength* of the groove modulation. It would later be shown by Löfgren in 1938 that tracking distortion is indeed proportional to this ratio.

The Löfgren 1938 Article – Weighted Tracking Error Minimisation

The formal relationship between the parameters linking tracking error and tracking distortion remained hidden until the publishing in November 1938 of the historic article by Professor Erik Olof Löfgren of the Royal Institute of Technology in Stockholm, Sweden.

Löfgren's article is the earliest work known to the writer which gives an analytical treatment of tracking distortion and its origin and develops a new optimum alignment method to minimise it. Löfgren applied mathematical rigor to the distortion model developed by Olney and undertook a Fourier analysis on the model. The results confirm the relationship postulated by Olney, which translates into the tracking distortion being proportional to the tracking error and *inversely proportional to the groove radius*. The tracking error divided by the groove radius is known as the Weighted Tracking Error (WTE). Löfgren then sought to minimise the tracking distortion by minimising the WTE. Löfgren developed three alignment strategies to meet three different situations. These are briefly expanded on below.

The 'Löfgren A' Alignment

This is Löfgren's **prime** alignment strategy for use when the offset angle and overhang are adjustable. *The offset angle and overhang are adjusted to minimise (and equalise) the three WTE peaks* across the recorded surface. This alignment is unique, as no other offset angle and overhang pair produces this alignment (for the arm length), as shown in [16]. Löfgren developed an optimisation method which involved applying the minimax principle (as used by Wilson) to the WTE. The maximum level of the distortion is then represented by the slope of the tracking error graph rather than by the level of the tracking error. The 'Löfgren A' alignment results in less tracking error at the inner grooves where the wavelengths are shorter. The introduction of this inverse radius weighting complicates the analytical solution, and Löfgren uses an approximation method, the error angle being small.

Löfgren's three-point, equal-WTE solution has continued to be applied to the present day.

The 'Löfgren B' Alignment

This is Löfgren's **compromise** alignment strategy for use when the overhang is adjustable but the *offset angle is fixed and non-optimum* (for the arm length). On page 360 of Löfgren's article, *Section 7: Discussion of configurations other than the optimum,* Löfgren considers those non-optimum, compromise situations, and writes: "…*Furthermore, there arises in practice the case where one wants to use a particular pickup with a non-optimum angular offset, while having flexibility concerning the mounting of the pickup relative to the platter. The question arises, which arrangement is the most favourable regarding distortion?*

"For the evaluation of different configurations regarding distortion, the maximum distortion parameters [the WTE peaks] *are not very suitable because for each case, it would be necessary to indicate three numbers: the two end maxima and the mid-point maximum. Since these maxima are not of equal importance* [refer 'Löfgren C' alignment below] *it is not sufficient to indicate only the largest of them. A more suitable quality measure is provided by the effective* [LMS] *distortion parameter* [EQN (44)]*"*. (The RMS distortion figure is also a suitable measure.)

Continuing to page 361: "*For an arbitrarily selected linear offset p, we want to determine that value of overhang g for which the effective distortion parameter K_{eff}* [EQN (44)] *is a minimum"*. Löfgren develops the required overhang equations at EQNs (47) and (48) for the alignment in this compromise situation. *For the specified, fixed, non-optimum offset angle,* the overhang is adjusted to minimise the LMS/RMS tracking distortion. (More follows.)

The 'Löfgren B' alignment considers an additional factor totally different to minimising WTE and tracking distortion across the playing surface, namely, minimising the LMS/RMS distortion in this compromise, non-optimum circumstance.

The 'Löfgren C' Alignment
This is Löfgren's **alternative** alignment strategy. The *offset angle and overhang are adjusted to minimise the LMS/RMS tracking distortion*. This alignment is also unique, as no other offset angle and overhang pair produces this alignment (for the arm length).

On page 359 of Löfgren's article, Löfgren considers that an objection could be levied against the magnitude of the central WTE peak of the 'Löfgren A' alignment. The 'Löfgren C' alignment is an attempt by Löfgren to ease this concern by reducing the magnitude of the central WTE peak of the 'Löfgren A' alignment by minimising the LMS/RMS distortion via adjustment of the offset angle and overhang.

"An objection that could be raised against the ['Löfgren A'] *calculations is that the three maximum values of the parameter δ / r* [WTE] *are not of the same importance. A greater importance should actually be attached to the maximum at r^* rather than the maxima at the inner and outer recorded radii r_1 and r_2, first because δ / r changes only slowly in the vicinity of r^*, while in contrast δ / r changes very rapidly at r_1 and r_2. Secondly, the inner and outer radii r_1 and r_2 are not necessarily utilised with each record. Because of this consideration, one should permit somewhat larger values of δ / r at r_1 and r_2 than at r^*".*

In considering the central WTE peak, at the last paragraph on page 359, Löfgren states: "*The greater importance of the maximum at r^* can be taken into account through the use of the method of least squares*", but then gives no further explanation of this. Löfgren presents an integral which is the square of the WTE, which he states "*…has to be minimised when varying D and φ_0 (or g and p respectively)"*, i.e., by varying the offset angle and overhang. (This integral *K* is also implicit in EQN (44) on the following page of his article.) By minimising *either* the page 359 integral or EQN (44) through adjustment of the offset angle and overhang, the *LMS/RMS tracking distortion will be reduced to the lowest possible value*, and so establish the required offset angle and overhang to achieve the 'Löfgren C' alignment.

Continuing at the top of page 360, Löfgren states: *"In accordance with this* [integral-minimising] *method, we will introduce an effective distortion parameter* [EQN (44)]*. It shows that the design for its smallest value departs only slightly from the design indicated above* [the 'Löfgren A' design]*. This design is, however, much more laborious"*. Löfgren is indicating the offset angle and overhang *values* required to minimise the LMS distortion (the 'Löfgren C' alignment) are only slightly different to the values required for the 'Löfgren A' alignment, but the mathematical procedure to calculate the offset angle and overhang *formulas* is much more complex ("laborious"), and so Löfgren only provides a formula for the overhang. He **does not provide** a formula for the offset angle, nor is any further explanation provided. (We can calculate the required offset angle today using Microsoft Excel. More follows.) By minimising the LMS/RMS distortion to the lowest possible level to achieve the 'Löfgren C' alignment, the magnitude of the central WTE peak is reduced, which Löfgren believes should mollify any possible objection to the 'Löfgren A' alignment.

The 'Löfgren C' alignment considers an additional factor totally different to minimising the WTE (and tracking distortion) across the playing surface. Löfgren felt some objection could arise regarding the relatively slowly changing WTE (and distortion) around the central WTE peak. This concern is addressed by this alternative alignment strategy which results in a reduction in the magnitude of the central WTE peak of the 'Löfgren A' alignment.

Löfgren's Fig. 4
Löfgren continues: *"In order to obtain an illustrative picture of how this parameter K_{eff} is dependent on the configuration, the author undertook numerical calculations for an average arm length, R = 22 cm and different combinations of linear offset p and overhang g"* to determine the resulting effective or LMS distortion. "*The distortion parameter K was then calculated according to the approximate formula $K \doteqdot V / \Omega \cdot \delta / r$* from EQN (22)*, with the assumption that V = 10 cm/s and $\Omega = 2\pi \cdot 78/60$ radians per second; δ was calculated according to Equations (1) and (2). Using the calculated values, curves of equal effective distortion parameter K_{eff} are drawn, with the axes being p and g in Fig. 4. The curves I-IV correspond to K_{eff} values of 0.5, 1, 1.5 and 2%.*

The dotted line in Fig. 4 is a line formed by *all* the intersecting points which provide *minimum LMS distortion* over the range of linear offset and overhang values shown. For information, point *A* in Fig. 4 is the point of minimum tracking error ala Wilson's alignment, point *B* is the point of minimum WTE (minimum *K)* which is the 'Löfgren A' alignment point and point C is the point of lowest effective or LMS distortion, K_{eff}, which is the 'Löfgren C' point, which in this case amounts to 0.25% distortion and is a *unique* point on the dotted line. *All other points on the dotted line* represent 'Löfgren B' alignment points at which the LMS distortion is *always* greater than that at point C. It may be noted that the LMS distortion for the 'Löfgren A' alignment (point B) is only slightly higher than that for the 'Löfgren C' alignment (point C). Furthermore, the offset angle-overhang values for the 'Löfgren A' point are only slightly different to those for the 'Löfgren C' point, as Löfgren reports.

Sensitivity to Alignment Errors

At the last paragraph on page 360, Löfgren states: *"From the shape and location of the curves in Fig. 4, it results that the largest distortion risk occurs when the overhang is not correctly set for the linear offset. On the other hand, the angular offset itself is not so critical"*.

We can demonstrate this as follows, noting that the curves in Fig. 4 are plots of points of equal LMS distortion. Consider point C (the 'Löfgren C' point), which has the lowest LMS distortion (0.25%) of *all* points in the Fig. 4 data space, and where *p* (linear offset) is approx. 8.3 cm and *g* (overhang) is approx. 1.5 cm. Starting at point C, if we say decrease *p* by moving horizontally to the left to meet the 2% distortion curve, *p* has changed from approx. 8.3 cm at point C to approx. 5.4 cm at the intersection with the 2% distortion curve, which is a change in *p* of approx. 2.9 cm. Starting again at point C, if we say increase *g* by moving vertically upwards to meet the 2% distortion curve, *g* has changed from approx. 1.5 cm at point C to approx. 2.5 cm at the 2% distortion curve intersection, which is a change in *g* of approx. 1.0 cm. Thus, *p* needs to be in error by 2.9 cm to cause the LMS distortion to increase from 0.25% to 2%, whereas *g* needs to be in error by only 1.0 cm to cause the same distortion increase. Thus, the increase in the LMS distortion resulting from alignment errors is more sensitive to errors in overhang *g* than to errors in linear offset *p* as Löfgren states.

For Löfgren's example using a 22 cm tonearm, instead of using Fig. 4 to determine the overhang *g* to use with a non-optimum linear offset *p* for the 'Löfgren B' alignment, a quick method to find *g* is given by the following least squares fit solution:

$$g = 0.003131 * p^2 + 0.3424 * p - 1.527 \qquad \text{(with } g \text{ and } p \text{ in cm)}$$

The Later Authors

In an independent, time-shifted, parallel development in 1940, Hume Huon [30] published an optimum alignment solution which minimises tracking error, identical to Wilson's 1924 solution. The existence of this paper was brought to the writer's attention in 2023 by Huon's son.

Löfgren's paper was followed by articles from Baerwald [2] in 1941, Bauer [3] in 1945, Seagrave [4] in 1956, Stevenson [5] in 1966, and Kessler and Pisha [6] in 1980.

Comparison of Major Articles

To the best of the writer's knowledge, no attempt has been made to analyse and compare the optimum design solutions provided in these six articles for the purpose of determining the historical facts and any similarities or differences between the articles. The writer investigated these aspects and found the results astounding!

The Simple Mathematical Fact

The optimum alignment equations for offset angle and overhang presented by Baerwald, Seagrave, Stevenson and Kessler and Pisha are *mathematically identical* to Löfgren's and differ only in notation and arrangement. One can only speculate as to why the later researchers devoted time and effort into repeatedly "solving" the alignment optimisation problem while knowing their equations are mathematically identical to those that went before!

The mathematical equivalence of these works was publicly exposed in 1983 by the writer [7] and is presented in detail later in this book.

The optimum alignment equations presented by Bauer are approximations to Löfgren's equations.

Stevenson's Solutions
Stevenson presented two alignment solutions. His alternative solution, 'Stevenson B', is *identical* to the 'Löfgren A' solution. His main solution, 'Stevenson A', is based on an algebraic rearrangement of the 'Löfgren A' equations to enable the *inner null radius* to be specified as an input parameter in lieu of the *inner groove radius*. The 'Stevenson A' strategy is to place the inner null radius at the inner groove radius to achieve *zero* WTE at the inner groove.

On page 216 of Part 1 of Stevenson's article, he writes: "*Most forms of distortion increase as the pickup approaches the turntable centre, so it is desirable for the distortion due to lateral tracking error to be very small at the average (inner groove radius)*".

Stevenson was concerned with the presence of the various types of distortion already existing at the inner grooves, especially tracing distortion, which results when the cutter and stylus are of different shapes. In the 'Stevenson A' alignment, the magnitudes of the WTE at R_W and R_2 are equalised.

Although the 'Stevenson A' alignment places the inner null radius (zero WTE) at the specified inner groove, the consequence is that the 'Stevenson A' WTE (and distortion) is greater than the 'Löfgren A' WTE (and distortion) over about 75 percent of the record playing surface.

Thus, Stevenson applies the 'Löfgren A' equations in two different ways, to achieve two different optimisations based on different goals.

The 'Stevenson A' alignment addresses an additional factor totally different to minimising the WTE across the playing surface, namely, the need to minimise distortion from *all* sources at the inner groove area.

To summarise, Stevenson's main approach, 'Stevenson A', provides a totally different outcome to Löfgren's by setting the WTE at the inner recorded groove area to zero, whereas Stevenson's alternative approach, 'Stevenson B', is identical to the 'Löfgren A' solution.

Some earlier commercial advertisements state that Stevenson's equations are more accurate than Baerwald's, which are "outdated". Based on the findings, this writer would argue that any such differences lie simply in the criteria used for selecting the inner recorded groove radius, and *not* in the underlying mathematics of the two articles. There are *no* differences there!

Summary of Facts

1. Harsanyi was probably the first to identify and apply offset angle and overhang principles to the pivoted tonearm to minimise tracking error and record wear.
2. Löfgren was the first to show that the distortion resulting from tracking error is primarily harmonic distortion and mainly second harmonic in nature. It is not only proportional to the tracking error but is also inversely proportional to the groove radius.
3. Löfgren was the first to develop optimum alignment equations for the offset angle and overhang, based on the three-point, equal-WTE solution ('Löfgren A').
4. Baerwald, Seagrave, Stevenson and Kessler and Pisha also produced optimum offset angle and overhang equations based on the three-point, equal-WTE solution. Their equations are *identical* to Löfgren's.
5. Löfgren, Baerwald and Bauer also produced *identical, but approximate*, equations for optimum overhang.
6. Seagrave also produced an approximation for optimum overhang which is more accurate than in 5 above.
7. Bauer's optimum offset angle equation was an approximation.
8. Löfgren also developed a compromise alignment ('Löfgren B') to accommodate the non-ideal situations where the offset angle is fixed and non-optimum for the arm length. This method adjusts the overhang to minimise the LMS/RMS tracking distortion.
9. Löfgren also developed an alternative alignment ('Löfgren C') to address possible conjecture that the three WTE peaks of the 'Löfgren A' alignment are not of equal importance. This approach minimises the LMS/RMS distortion, which has the effect of lowering the WTE peak between the null radii, but at the expense of increasing the WTE at the inner and outer grooves. Löfgren does not provide offset angle and overhang formulas.
10. Stevenson's main approach ('Stevenson A') applies the 'Löfgren A' equations in a different manner, to achieve a different alignment outcome, in which the inner null radius is placed at the inner groove radius to minimise distortion from WTE at the inner grooves. The consequence of this alignment is an *increase in WTE over most* (approx. 75 percent) of the playing surface compared to the 'Löfgren A' alignment.

11. Stevenson's alternative approach ('Stevenson B') is identical to Löfgren's main approach ('Löfgren A').
12. The 'Stevenson A' alignment is opposite in psycho-acoustic terms to the 'Löfgren B' alignment, in so far as the distortion between the null radii is concerned, because the 'Stevenson A' alignment increases it, whereas the 'Löfgren B' alignment decreases it.
13. Seagrave states that the offset angle and overhang equations provided by Bauer can result in errors of up to one degree in offset angle and five percent in overhang, due to the mathematical approximations used by Bauer.

Conclusion
The three-point, equal-WTE optimisation by Löfgren in 1938 has provably *not* been outdated by the later authors. The fact that Baerwald, Seagrave, Stevenson and Kessler and Pisha produced optimum design equations *identical* to Löfgren's confirms the relevance, the preciseness and the historical significance of Löfgren's work. The articles of Löfgren and Baerwald are the definitive references on the subject of tracking distortion, its analysis and its reduction. Baerwald also provided additional insight into the problems of achieving optimum alignment and ways to minimise the sensitivity of the alignment to error conditions.

Further, it should be stated loud and clear that if Löfgren's article had been the only article ever published on the subject of optimum tonearm alignment for the reduction of WTE, we would *still* be using the same optimum design equations we use today.

Today, the selection of an appropriate inner groove radius is the remaining challenge we face, not the selection of the optimum design formulae to use. For the optimum design formulae, we have the choice between Löfgren's, Baerwald's, Seagrave's, Stevenson's and Kessler and Pisha's!

Finally, we should recognise the contributions to the field of optimum tonearm alignment geometry provided by these audio researchers:
B. Harsanyi, L. Lumière, P. Wilson, E.G. Löfgren, H.G. Baerwald, B.B. Bauer, J.D. Seagrave, J.K. Stevenson, M.D. Kessler and B.V. Pisha.

New Zero Radii Formula for Wilson's Alignment
A newly-developed formula for calculating the zero tracking error (null) radii for Wilson's alignment, based on tracking error minimisation, has been developed by Vladan Jovanovic [17]. It uses the inner and outer groove radii only. This is the first formula for the purpose known to the writer. The formula is shown on page S1-63 in the Alignment for Tracking Error Minimisation section.

Historic 'Löfgren C' Offset Angle Formula
Löfgren did not provide a solution to implement the 'Löfgren C' alignment strategy, as already noted. A recent paper by Vladan Jovanovic [18] presents a solution for the required linear offset *p*, from which the offset angle β may be readily determined, enabling the 'Löfgren C' alignment to be implemented using formulas. To summarise, when Jovanovic's distortion-minimising linear offset formula is used in conjunction with Löfgren's 'Löfgren B' distortion-minimising overhang formula at his EQNs (47) and (48), the 'Löfgren C' *formula-based* distortion-minimising alignment may finally be implemented!! Refer to page S1-24 and page S9-22 for the details.

Historic Analytical Solution To Löfgren's Alignment Equations
In a recent paper by Peet Hickman [19], the offset angle and overhang formulas for the 'Löfgren A', 'Löfgren B' and 'Löfgren C' alignments are fully derived from first principles. This is an historic first. This includes the linear offset formula for the 'Löfgren C' alignment, which was first provided by Jovanovic [18]. Hickman proves the equivalence of Jovanovic's linear offset formula with his own. Refer to page S9-23 for Hickman's linear offset formula for the 'Löfgren C' alignment.

NOTATION USED IN THIS BOOK

To assist the reader's understanding, the notation used in the analyses of the six major articles (Sections S2 to S7) is the same as that used by the respective original authors. A common notation is used for the remainder of the book and is listed on page S9-1.

REFERENCES

Note that many of the following references are available on the Vinyl Engine web site.

1. Löfgren, Erik, "Über die nichtlineare Verzerrung bei der Wiedergabe von Schallplatten infolge Winkelabweichungen des Abtastorgans", *Akustische Zeitschrift* (German), November 1938, pg. 350.
The German-language article may be accessed at:
https://www.vinylengine.com/turntable_forum/viewtopic.php?f=46&t=11326 then click Download.
A translation of this article was written by Klaus Rampelmann in 2008 and is titled:
"On The Non-Linear Distortion In The Reproduction Of Phonograph Records Caused By Angular Deviation Of The Pick-up Needle". The translation may be accessed at:
https://www.vinylengine.com/turntable_forum/viewtopic.php?t=15875 then click Download.
2. Baerwald, H.G., "Analytic Treatment Of Tracking Error And Notes On Optimal Pickup Design", *Journal of Society of Motion Picture Engineers*, December 1941, pg. 591.
3. Bauer, B.B., "Tracking Angle In Phonograph Pickups", *Electronics*, March 1945, pg. 110.
4. Seagrave, J.D., "Minimising Pickup Tracking Error", *Audiocraft Magazine*, Part 1, December 1956, pg. 19; Part 2, January 1957, pg. 25; Part 3, August 1957, pg. 22.
5. Stevenson, J.K., "Pickup Arm Design", *Wireless World*, Part 1, May 1966, pg. 214; Part 2, June 1966, pg. 314; "Addenda", August 1966, pg. 396.
6. Kessler, M.D. and B.V. Pisha, "Tonearm Geometry and Setup Demystified", *Audio*, January 1980, pg. 76; "Addenda", April 1980, pg. 26.
7. Dennes, Graeme F., "Signals & Noise" Letter, Audio, May 1983, pg. 48.
8. Wilson, Percy, "Needle Track Alignment", *The Gramophone*, Part 1, September 1924, pg. 129; Part 2, October 1924, pg. 167.
9. Harsanyi, Bela, French Patent 385526, filed 24 December 1907, granted 10 March 1908, published 15 May 1908. The French-language patent may be accessed at:
https://worldwide.espacenet.com/publicationDetails/originalDocument?CC=FR&NR=385526A&KC=A&FT=D&ND=3&date=19080515&DB=EPODOC&locale=en_EP# then click Download.
10. Lumière, Louis, French Patent 422638, filed 24 January 1910, granted 24 January 1911, published 27 March 1911. The French-language patent may be accessed at:
https://worldwide.espacenet.com/publicationDetails/originalDocument?CC=FR&NR=422638A&KC=A&FT=D&ND=3&date=19110327&DB=EPODOC&locale=en_EP# then click Download.
11. Löfgren, Erik, "Nålföringen vid grammofoner", *Radio och Grammofon* (Swedish), vol. 7, no. 19, pgs. 15-18, 24 and vol. 7, no. 22-23, pgs. 10-14. Published 1929. A translation of this article was written by Klaus Rampelmann in 2021 and is titled: "Gramophone Needle Alignment". The translation may be accessed at:
https://www.vinylengine.com/turntable_forum/viewtopic.php?f=46&t=125463 then click Download.
12. Howard, Keith, "Arc Angles: Optimizing Tonearm Geometry", *Stereophile Magazine*, April 2010. The document may be accessed at:
https://www.stereophile.com/features/arc_angles_optimizing_tonearm_geometry/index.html
13. Wilson, Percy, "Two Notes On Gramophone Adjustment", *The Gramophone*, March 1925, pg. 381.
14. Bird, J.R. and C.M. Chorpening, "The Offset-Head Crystal Pickup", *Radio Engineering*, March 1937, pg. 16.
15. Olney, B., "Phonograph Pickup Tracking Error vs Distortion And Record Wear", *Electronics*, November 1937, pg.19.
16. Dennes, Graeme F., "Is The 'Löfgren A' Solution Unique?", June 2010. The document may be accessed at: https://www.vinylengine.com/turntable_forum/viewtopic.php?f=46&t=19199 then click Download.
17. Jovanovic, Vladan, "Zeros of the Tracking Error for Wilson's Optimal Tonearm Alignment", June 2021. The document may be accessed at: https://www.vinylengine.com/turntable_forum/viewtopic.php?f=46&t=126024 then click Download.
18. Jovanovic, Vladan, "New Analytical Results for *Löfgren C* Tonearm Alignment", *Journal of Audio Engineering Society*, March 2022, pg. 199. The document may be accessed at:
https://www.aes.org/e-lib/browse.cfm?elib=21560 then click Download Now.
19. Hickman, A. P., "Analysis of Löfgren's Tonearm Optimization", *Journal of Audio Engineering Society*, January/February 2023, pg. 53. The document may be accessed at:
https://secure.aes.org/forum/pubs/journal/?elib=22031
20. Bauer, B.B., "Pickup Placement", *Electronics*, June 1949, pg. 87.
21 Carlson, R.E., "Resonance, Tracking And Distortion – An Analysis Of Phonograph Pickup Arms", *Journal of Audio Engineering Society*, July 1954, pg. 151. Also reprinted in Audio Engineering Society Anthology Series, Disk Recording Vol 2 - Disk Playback and Testing, pg. 259.
22. Gilson, R.J., "The Cartridge Alignment Problem", *Wireless World*, October 1981, pg. 59.
23. Mohr, David P., "A History Of The Study Of Lateral Tracking Geometry And Its Effects On Playback", presented at the 74th Audio Engineering Society Convention, New York, 12 October 1983.
24. Randhawa, T.S., "Pickup Arm Design Techniques", *Wireless World*, Part 1, March 1978, pg. 73; Part 2, April 1978, pg. 63.
25. *The Audio Critic*, Vol. 1, Nos. 1, 4, 5 and 6.
26 *The Audio Critic*, "Cartridge, Arm and Turntable vs. the Groove: Who's Winning?", December 1977, pg. 49.
27. *The Audio Critic*, "Lateral Tracking Alignment Revisited (Maybe Your Overhang Is Wrong After All)", 1987, pg. 6.
28. Tollerton, Richard, "Digital Simulation of Phonograph Tracking Distortion", presented at the 127th Audio Engineering Society Convention, New York, 9 October 2009.
29. Wilson, P. and G.W. Webb, Modern Gramophones and Electrical Reproducers, pg. 121-132 (Cassell and Company, London), 1929.
30. Huon, H.F., "The Cranked Pick-Up Tone Arm", *Proceedings of the Institution of Radio Engineers Australia*, Vol. 4, No. 2, May 1940, pg. 26.

GENESIS OF THIS BOOK

The first release of this book, *sans* Löfgren, was written by the writer in 1982.

A copy of the book was forwarded to the Editor of *Wireless World*, along with a "Letter". The editor agreed to publish the Letter and did so in the February 1983 edition. The conclusions were that, to the best of the writer's knowledge, Baerwald was the first to produce the optimum alignment formulas, and that the formulas from the later authors were identical to his.

[To that time, I had not sighted Erik Löfgren's 1938 article, nor was I aware of its contents or significance. My only reference to it was in Baerwald's article, in which he refers to Löfgren's article three times. I would normally not include an unsighted reference, but as no one else had referred to or cited Löfgren's article as far as I was aware, and as Baerwald was a well-published and respected engineer, I decided to include the reference to Löfgren's article in the hope that someone may have it or know where to locate it. I also used Baerwald's English translation of the title of Löfgren's article.]

Around two weeks after the *Wireless World* letter was published, I received a letter from Dr Barney V. Pisha, Associate Editor at *Audio*. He saw the *Wireless World* Letter with my reference to Löfgren and said he had been searching for a copy of Löfgren's article for three years. He asked me if I had a copy of it, and when I said I did not, he said he would continue searching for it. He contacted me two weeks later and said he had located a person in Czechoslovakia who had a copy of the article, and he would send me a copy once he received it. When I did receive the copy from Dr Pisha about three weeks later, I immediately reviewed the mathematics in it and realised that Löfgren had produced the optimised three-point equal WTE solution three years before Baerwald! I was ecstatic. This changed the historical facts known to (almost) the entire audio world. I contacted Barney Pisha and gave him the news of my findings. It also meant that the findings I had presented in the Letter to *Wireless World*, just published, and in my analysis book, had now been superseded by way of this new information.

Dr Pisha suggested that if I wished to prepare a "Letter to the Editor" about these new findings, *Audio* would publish it. After revising my tonearm analysis book to include the results of Löfgren's work, I forwarded a Letter and the revised book to *Audio*. The Letter was published in the May 1983 edition [7].

The Letter in *Audio* created far greater and wider interest than this writer could ever have imagined. Several dozen letters were received by the writer requesting a copy of the analysis book. These were mainly from audio equipment manufacturers across several countries (many of which were industry names in the turntable, tonearm and cartridge fields), university academics and researchers, and other well-known names in the audio industry and audio journals.

In summary, we now know the *true* historical facts about the origin and underlying cause of tracking distortion, and how we can minimise it through the application of the results and methods presented by Löfgren. My grateful appreciation goes to Dr Pisha for his efforts in locating Löfgren's article and for making a copy available to me.

As background, Dr Barney Pisha commanded great respect from people in numerous fields. He was a Doctor of Medicine and practicing physician. Because of his expertise in clinical pharmacology, he was instrumental in the development of several significant drug modifications which entered into wide use. He also possessed formal music and electrical engineering qualifications and was a licensed radio amateur. He was Associate Editor at *Audio* and the owner of an audio research laboratory for testing all things audio. He had countless global contacts in the audio industry. He was truly an extraordinary man.

I was privileged to meet him in person at the 74th AES Convention in New York in October 1983. Dr Pisha was made a Fellow of the Audio Engineering Society in 1986.

Sadly, Dr Barney V. Pisha passed away on 6 November 1991. (*J. Audio Eng. Soc.*, Jan/Feb 1992.)

GRAEME F. DENNES

GROOVE RADII SELECTION

Although standards, quasi-standards and popular values for inner and outer groove radii have been presented and used in the past, any such values are *only* relevant for optimum alignment purposes for those records which match them. While the outer recorded groove radii have generally shown some consistency in the past, the inner recorded groove radii have generally shown little consistency, adding to the complexity of optimum arm alignment for the audiophile and the tonearm and turntable manufacturers.

Consequently, selecting groove radii values for optimum alignment purposes based upon any one standard is problematic in that they may have little intrinsic relevance to the groove radii values actually measured in *your* record collection. Further, the use of groove radii values based on someone else's record collection is equally problematic for the same reason.

Realistically, we should measure the groove radii within our own record collections, and then make suitable judgements in selecting the radii to use with the optimum design formulae.

What we really need is a *set of criteria* for selecting suitable groove radii, not just numbers! There are no shortage of numbers about! We need a procedure – a methodology – which, based upon some property or parameters, will lead us to the groove radii values to use in the design equations for our own purposes. Of course, it will probably be statistically based, but at least it will apply to *our* record collection!

As an example only: In an attempt to obtain a reliable basis for calculations, Fig. 5 on the last page of Löfgren's article shows graphs for the inner radii of a large number of 25 cm and 30 cm records. He also describes a method of assessing the data for selecting the inner groove radius for use in the calculations.

SUMMARY OF ALIGNMENT STRATEGIES AND CALCULATIONS

Four different alignment strategies are discussed in the articles analysed, as follows:

1. **The 'Löfgren A' strategy:** Minimises the WTE (and tracking distortion) by equalising and minimising the magnitudes of the three WTE peaks at R_1, R_W and R_2 through adjustment of the offset angle and overhang. This is Löfgren's prime alignment strategy.

   The 'Löfgren A' solution: Löfgren provides formulas for the optimum offset angle and overhang to implement this strategy.

   This alignment strategy and solution is unique to Löfgren.

   The alignment strategies and solutions from Baerwald (1941), Bauer (1945), Seagrave (1956) and Kessler and Pisha (1980) are identical to those of Löfgren as the writer advised in 1983.

2. **The 'Löfgren B' strategy:** Minimises the LMS/RMS tracking distortion between R_1 and R_2 through adjustment of the overhang for when the offset angle is fixed and non-optimum. This is Löfgren's compromise alignment strategy in this non-ideal situation.

   The 'Löfgren B' solution: Löfgren provides a distortion-minimising optimum overhang formula at EQNs (47) and (48) which minimises the LMS/RMS distortion to implement this strategy.

   This alignment strategy and solution is unique to Löfgren.

3. **The 'Löfgren C' strategy:** Minimises the LMS/RMS tracking distortion between R_1 and R_2 through adjustment of the offset angle and overhang. This is Löfgren's alternative alignment strategy (to address possible objections to the 'Löfgren A' alignment's central WTE and tracking distortion peak).

   The 'Löfgren C' solution: Löfgren does not provide a solution. Although his 'Löfgren B' distortion-minimising overhang formula, which minimises LMS/RMS distortion, may correctly be applied to the 'Löfgren C' case, no formula for the required offset angle existed between 1938 and 2021.

   Vladan Jovanovic [18] developed a formula for the 'Löfgren C' linear offset, from which the 'Löfgren C' offset angle formula may be readily determined. When this offset angle formula is used in conjunction with Löfgren's 'Löfgren B' overhang formula, we have a highly accurate, formula-based, distortion-minimising solution for the 'Löfgren C' alignment strategy - an historic first!

   This alignment strategy is unique to Löfgren. The solution is unique to Löfgren and Jovanovic.

4. **The 'Stevenson A' strategy:** Places the inner zero-error radius R_{01} at the inner record groove radius and equalises the magnitudes of the WTE at R_W and R_2 through adjustment of the offset angle and overhang.

   The 'Stevenson A' solution: Stevenson provides formulas for the optimum offset angle and overhang to implement the strategy, based on a re-arrangement of Löfgren's 'Löfgren A' formulas to enable the zero-error radius R_{01} to be specified as an input.

   This alignment strategy is unique to Stevenson and the solution is unique to Löfgren and Stevenson.

HISTORY OF TONEARM ALIGNMENT SOLUTIONS

PRE-1908:
The writer is not aware of tonearm alignment optimisation being considered nor the principles of offset angle and overhang being applied before Harsanyi's 1908 patent.

1908 to 1929:
This was the period when the principles of offset angle and overhang were being applied to minimise tracking distortion and record wear by minimising *tracking error*. The works of Harsanyi in 1908, Lumière in 1911, Wilson in 1924 and Löfgren in 1929 were fundamental to the development of optimum alignment formulas to minimise tracking error.

1938:
This was the year Löfgren published his ground-breaking article [1] on optimum tonearm alignment which showed that *weighted* tracking error, not tracking error, needed to be minimised to reduce distortion to the lowest extent possible. Löfgren presented three alignment strategies to optimise tonearm alignment for three different situations. Löfgren provided the offset angle and overhang formulas for the "Löfgren A" alignment and the overhang formula for the "Löfgren B" and the "Löfgren C" alignment.

1941 to 1980:
In the years 1941, 1945, 1956, 1966 and 1980, the latter five of the six articles examined by the writer in 1983 were published [2] to [6].

1983:
Via a letter to *Audio* in 1983 [7], the writer showed that the alignment formulas presented in the six major articles on tonearm alignment [1] to [6] were identical apart from notation and arrangement. This realisation formed the genesis of the first version of this book which showed that the latter five articles presented alignment formulas *identical* to those of Löfgren in 1938.

2010:
In the search for greater accuracy, two enhancements were made by the writer to the 2010 release of this book. Firstly, the concept of the "perfect" or high-accuracy alignment solutions were introduced, using the Microsoft Excel Solver tool to greatly increase the accuracies of the calculated offset angle and overhang values as replacements for Löfgren's formulas. This overcame the mathematical approximations inherent in the alignment formulas. The "perfect" solution method was not only applicable to the 'Löfgren A', 'Löfgren B' and 'Stevenson A' alignments, but also to the 'Löfgren C' alignment and was in fact the only way to obtain the optimum offset angle setting to implement the 'Löfgren C' alignment. High accuracy solutions could at last be obtained for all four alignment strategies discussed in this book.

Secondly, the provision of a spreadsheet-based RMS distortion calculator based on the Composite Trapezoidal Rule numerical integrator was introduced. This enabled the RMS distortion resulting from a given set of alignment parameters to be calculated for any alignment situation. This RMS distortion tool greatly improved on the accuracy of the distortion formula presented by Löfgren's EQN (46).

2021:
Löfgren's offset angle and overhang formulas are based on his distortion term ε at EQN (22), the fundamental equation unifying the four key variables: maximum peak recorded velocity V, turntable speed Ω, tracking error δ and groove radius R. Löfgren's distortion formula at his EQN (46) is derived from his EQN (22). However, EQN (22) is an approximation in that it uses tracking error δ (in radians) as an approximation to TAN(δ). This approximation was used by Löfgren to simplify the development of the alignment formulas.

Although an acceptable approximation for small δ, EQN(22) results in small errors being introduced into Löfgren's offset angle and overhang formulas. It also meant that the "perfect" solutions using the Microsoft Excel Solver tool, introduced by the writer in 2010, also carried some inaccuracies because they too were based on Löfgren's EQN (22).

As background, Löfgren's distortion term ε at equation EQN (22) is:

$$\varepsilon = V / \Omega * ABS(WTE)$$

where $WTE = \delta / R$

where V = maximum peak recorded velocity, Ω = record speed, δ = tracking error in radians and R = groove radius.

High Accuracy Distortion and WTE Formula:
In the development leading up to his EQN (22), Löfgren's theoretically correct (no-approximation) distortion term uses TAN(δ) in lieu of δ (radians), so in the search for even greater accuracy and new to the 2021 revision of this book, the writer has employed TAN(δ) in place of δ (radians) in the distortion term ε.

This means we replace Löfgren's distortion formula EQN (22) above with Löfgren's high accuracy distortion formula:

$$\varepsilon = V / \Omega * ABS(WTE)$$

where $WTE = TAN(\delta) / R$.

The high accuracy distortion formula and the high-accuracy WTE formula are also used throughout the writer's accompanying tonearm alignment spreadsheet, with the results provided in the pages following.

Because of this different but more accurate method to calculate the WTE, the results calculated may be slightly different to those calculated previously using Löfgren's EQN (22).

In summary, the four "perfect" offset angle and overhang solutions calculated using the methods and improvements described above provide the most accurate alignment solutions possible to meet the four alignment strategies.

Note on Terminology:
Löfgren proved that tracking distortion is proportional to the WTE, where the WTE is calculated by dividing the TE by the groove radius R, i.e., TE / R, where the TE is weighted by the inverse of the groove radius, as already discussed.

With the 2021 release of this book, Löfgren's high-accuracy distortion term above has replaced Löfgren's EQN (22) for all distortion calculations, so the WTE term is now calculated by dividing the *tangent* of the tracking error by the groove radius R, i.e., TAN(TE)/R, where the term TAN(TE) is weighted by the inverse of the groove radius. This means we are no longer dealing with weighted TE but rather with weighted TAN(TE). The "weighted" term is correct, but the term "TAN(TE)" is *not* the same as "TE", thus the term "weighted TE" is no longer correct! The correct term is "weighted TAN(TE)", but that does seem a handful…

WTE
Realising the term WTE (weighted tracking error) has been used for many decades in many articles, publications, books and communications, and realising its formula has been changed in this book through the adoption of Löfgren's high-accuracy distortion formula above, this writer has chosen to retain the general use of the term "WTE" in this book and its accompanying spreadsheet, knowing it to be a proxy for the accurate term, TAN(TE) / R.

2022:
Vladan Jovanovic [18] was the first to develop the linear offset formula for the 'Löfgren C' alignment, from which the offset angle is calculated, enabling this alignment to be implemented using formulas. Löfgren did not provide a formula for the offset angle. See page S9-22.

2023:
Peet Hickman [19] was the first to analytically derive the 'Löfgren A', 'Löfgren B' and 'Löfgren C' alignment formulas from first principles. He also proved that Jovanovic's linear offset formula for the 'Löfgren C' alignment [18] is equivalent to his own. This AES paper may arguably be the most significant contribution to tonearm alignment since Löfgren's 1938 paper. See page S9-23.

MICROSOFT EXCEL AND HIGH-ACCURACY CALCULATIONS

The ‘perfect’ alignment solutions provided by the Solver tool in Microsoft Excel are of greater accuracy than the solutions provided by the alignment formulas for those alignments. The purpose of the “perfect” solutions is not to provide high-accuracy, extended precision alignment values per se, but to provide insight into the techniques used to calculate these values. They also provide more accurate and practical alignment values which overcome the minor inaccuracies inherent in the formulas which result from the approximations used by their respective authors. The background to the high accuracy techniques is described on pages S9-10 and S12-15.

The high-accuracy or 'perfect' alignment solutions are calculated using an iterative optimisation process managed by Excel’s Solver tool. The results you obtain may vary slightly from those shown in the solutions following. Because Solver is an iterative, algorithm-driven optimisation tool with finite bounds on its capabilities, its results can sometimes be influenced by the initial values of the changing cells, and by the values of the variables being used. Often, when the Solver finds a valid solution, if it is *immediately* run again, it can sometimes find a *slightly* different but still valid solution, after which further runs may not produce further change. Sometimes, the Convergence and Precision settings in the Solver may need to be larger to enable the Solver to find a valid solution or smaller to force it to seek a higher-accuracy solution. In principle, start with the settings shown, then try changing them up or down in small steps if and as required. Keep in mind that if you make them too small, the Solver tool may advise it cannot find a solution. You may also try varying the figures for the maximum number of iterations and the maximum time to ensure the Solver doesn’t find itself in an endless loop. Limiting the range of input parameters to realistic values also assists the Solver operation.

If the Solver reports: “The Objective Cell values do not converge”, *immediately* run it again to retry!

EXAMPLES OF ALIGNMENT CALCULATIONS

As already noted, this book is based on the analysis of four specific alignment strategies – three from Löfgren and one from Stevenson.

In this section, there are two sets of calculations and results for each of the four strategies. The first set is based on the respective formulas for the four strategies, while the second set is based on the application of the Excel Solver tool to the four strategies.

The eight alignment examples which follow use these parameters: arm length **= 250 mm**, inner recorded groove radius **= 60.325 mm** and outer recorded groove radius **= 146.05 mm**.

The total of eight alignment calculations presented are as follows:

a. ‘Löfgren A’ alignment using Löfgren's equations for offset angle and overhang to minimise the WTE.
b. ‘Perfect Löfgren A’ (high-accuracy) alignment using Microsoft Excel to calculate the offset angle and overhang for the 'Löfgren A' alignment to equalise and minimise the WTE to the *maximum extent possible*.
c. ‘Löfgren B’ alignment using Löfgren's equation for overhang to use with a fixed, non-optimum offset angle to minimise the LMS/RMS distortion.
d. ‘Perfect Löfgren B’ (high-accuracy) alignment using Microsoft Excel to calculate the overhang to use with the fixed, non-optimum offset angle for the 'Löfgren B' alignment to minimise the LMS/RMS distortion to the *maximum extent possible*.
e. ‘Löfgren C’ alignment based on a key formula developed by Vladan Jovanovic [18]. Löfgren did not provide a solution to the ‘Löfgren C’ alignment.
f. ‘Perfect Löfgren C’ (high-accuracy) alignment using Microsoft Excel to calculate the offset angle and overhang to minimise the LMS/RMS distortion to the *maximum extent possible*.
g. ‘Stevenson A’ alignment using Stevenson's equations to calculate the required offset angle and overhang to achieve zero WTE at the inner groove radius.
h. ‘Perfect Stevenson A’ (high-accuracy) alignment using Microsoft Excel to calculate the offset angle and overhang for the “Stevenson A’ alignment to achieve zero WTE at the inner groove radius to the *maximum extent possible*.

Exact, Universal, Alignment-Independent Formulas

These formulas are exact (except where noted) and are independent of the alignment method used. They may be universally applied.

$WTE = TAN (ASIN ((R^2 + L^2 - M^2) / (2 * L * R)) - \beta) / R$
$TA = ASIN((R^2 + L^2 - M^2) / (2 * L * R))$
$TE = TA - \beta$
$M = L - d$
$M = SQRT (L^2 - Ra^2)$
$L^2 - M^2 = 2 * L * d - d^2$
$Ra^2 = L^2 - M^2$
$Ra^2 = 2 * L * d - d^2$
$Ra^2 = R_{01} * R_{02}$
$d = L - SQRT (L^2 - Ra^2)$
$Rw = Ra^2 / p$ <<< Uses small angle approximations
$p = L * \sin \beta$
$p = (R_{01} + R_{02}) / 2$
$R_{01} = p - SQRT (p^2 - Ra^2)$
$R_{02} = p + SQRT (p^2 - Ra^2)$

In the formulas above, R = (some) radius, Ra = radius at the minimum of the angular tracking error curve, Rw = radius at the minimum of the WTE curve, L = tonearm length, d = overhang, M = mounting distance, p = linear offset, R_{01} = inner null radius, R_{02} = outer null radius. All these have units of length.
β = offset angle, TA = tracking angle, TE = tracking error. These have units of angle.

Ensure units are consistent, i.e., the same unit of length is used, such as mm, and the same unit of angle is used, such as degrees.
Note that **Excel's angular mode is radians**, so conversion to/from degrees is necessary when calculating in degrees in Excel.
To convert: radians = degrees * PI / 180, and degrees = radians * 180 / PI

Weighted Tracking Error at radius R
The following formulas use Löfgren's high-accuracy WTE expression, TAN(δ) / R. Prior to the 2021 revision of this document, the WTE expression used was Löfgren's approximation, δ (radians) / R, from his EQN (22).
The following equivalent formulas calculate the weighted tracking error (WTE) at radius R.

$WTE = TAN (ASIN ((R^2 + L^2 - M^2) / (2 * L * R)) - \beta) / R$ (Exact)
$WTE = TAN (ASIN ((R^2 + 2 * L * d - d^2) / (2 * L * R)) - \beta) / R$ (Exact)

Derivative of the Weighted Tracking Error (WTE) Function (TAN(δ)/R)
This (exact) formula calculates the derivative of the WTE function with respect to radius R. When used in conjunction with the Solver tool in the four 'perfect' solutions, it enables the radius Rw at the minimum of the WTE function to be calculated, which in turn enables the WTE at Rw to be calculated without approximation.

To calculate the derivative of the WTE function at radius R, use this single-line formula. The formula evaluates to zero when R is set to the value of Rw calculated by the Solver tool.

$=((1/L-(L^2-M^2+R^2)/(2*L*R^2))*SEC(\beta-ASIN((L^2-M^2+R^2)/(2*L*R)))^2)/(R*SQRT(1-(L^2-M^2+R^2)^2/(4*L^2*R^2)))+TAN(\beta-ASIN((L^2-M^2+R^2)/(2*L*R)))/R^2$

Derivative of the Tracking Error (TE) Function (δ)
This (exact) formula calculates the derivative of the TE function with respect to the radius R. The formula evaluates to zero when R is set to the value of Ra as given by the formulas above.

To calculate the derivative of the TE function at radius R, use this formula:
$=(R^2-L^2+M^2)/(2*L*R^2)/SQRT(1-(R^2+L^2-M^2)^2/(4*L^2*R^2))$

NOTE: Due to the operation of the Excel Solver optimising tool, your tonearm spreadsheet figures for the four main 'perfect' sheets may differ in the last few digits from those in the four 'perfect' examples following.

'LÖFGREN A' EXAMPLE USING FORMULAS

Purpose: To calculate the offset angle and overhang which minimise the WTE at the three radii R_1, R_W and R_2 by making the three peaks equal in magnitude.

Formulas derived by Löfgren, except for the null radii formulas which were first presented by Baerwald:

β	= DEGREES (ASIN ((R_1 * R_2 * (R_1 + R_2)) / L / ((R_1 + R_2)^2 / 4 + R_1 * R_2)))
d	= L - SQRT (L^2 - Ra^2)
Ra^2	= (2 * (R_1 * R_2)^2) / ((R_1 + R_2)^2 / 4 + R_1 * R_2)
p	= L * SIN (β)
M	= L - d
Rw	= (2 * R_1 * R_2) / (R_1 + R_2)
R_{01}	= (2 * R_1 * R_2) / ((1 - 1 / SQRT (2)) * R_1 + (1 + 1 / SQRT (2)) * R_2)
R_{02}	= (2 * R_1 * R_2) / ((1 + 1 / SQRT (2)) * R_1 + (1 - 1 / SQRT (2)) * R_2)
WTE R_1	= TAN (ASIN ((R_1^2 + 2 * L * d - d^2) / (2 * L * R_1)) - β * PI() / 180) / R_1
WTE Rw	= TAN (ASIN ((Rw^2 + 2 * L * d - d^2) / (2 * L * Rw)) - β * PI() / 180) / Rw
WTE R_2	= TAN (ASIN ((R_2^2 + 2 * L * d - d^2) / (2 * L * R_2)) - β * PI() / 180) / R_2

Inputs:

Tonearm length L, inner groove radius R_1, outer groove radius R_2.

Results:

Offset angle β	=	21.9488704000096 degrees
Overhang d	=	16.5018227481191 mm
Radius Rw	=	85.3830769230769 mm
d/dR of WTE at Rw	=	7.52860385795674E-09 TAN(TE) per mm^2
Radius Ra	=	89.3230161831161 mm
d/dR of TE at Ra	=	1.95276467860068E-18 TAN(TE) per mm
Radius Ra^2	=	7978.60122004922 mm^2
Inner null radius R_{01}	=	65.9980463741756 mm
Outer null radius R_{02}	=	120.891475708456 mm
Sine of offset angle β	=	0.373779044165264
Linear offset p	=	93.4447610413159 mm
Mounting distance M	=	233.498177251881 mm
WTE at R_1	=	0.000204114308016168 TAN(TE) per mm
WTE at Rw	=	-0.000202914713477743 TAN(TE) per mm
WTE at R_2	=	0.000204919107489487 TAN(TE) per mm
%Distortion at R_1	=	0.4340492801749900
%Distortion at R_W	=	0.4314983411890580
%Distortion at R_2	=	0.4357606870600560
WTE at R_1 - WTE at R_2	=	-8.04799473319307E-07 TAN(TE) per mm
WTE at R_1 + WTE at Rw	=	1.19959453842423E-06 TAN(TE) per mm
%RMS Distortion	=	0.287637896045523

'PERFECT LÖFGREN A' EXAMPLE USING MICROSOFT EXCEL SOLVER

Purpose: To calculate the offset angle and overhang required to minimise the WTE at the three radii R_1, R_W and R_2 by making the three peaks equal in magnitude.

Excel Solver settings:

Inputs:	Tonearm length L, inner groove radius R_1, outer groove radius R_2
Calculates:	The required offset angle and overhang
Objective cell:	WTE R_1 - WTE R_2
Equal to value of:	0
Changing cells:	Offset angle, overhang, radius Rw
Constraints:	1. The derivative of the WTE function at Rw = 0
	2. WTE R_1 + WTE Rw = 0
Solving method:	GRG Nonlinear
Derivatives:	Central
Precision:	1E-15
Convergence:	1E-15

Results:

Offset angle β	=	21.9629973051267 degrees
Overhang d	=	16.5157297049773 mm
Radius Rw	=	85.3885121356419 mm
d/dR of WTE at Rw	=	0.0 TAN(TE) per mm^2
Radius Ra	=	89.3593617076621 mm
d/dR of TE at Ra	=	0.0 TAN(TE) per mm
Radius Ra^2	=	7985.09552480080 mm^2
Inner null radius R_{01}	=	65.9789310351004 mm
Outer null radius R_{02}	=	121.0249302243590 mm
Sine of offset angle β	=	0.3740077225189180
Linear offset p	=	93.5019306297295 mm
Mounting distance M	=	233.484270295023 mm
WTE at R_1	=	0.000203894829749618 TAN(TE) per mm
WTE at Rw	=	-0.000203894829749615 TAN(TE) per mm
WTE at R_2	=	0.000203894829749618 TAN(TE) per mm
%Distortion at R_1	=	0.4335825594216260
%Distortion at Rw	=	0.4335825594216180
%Distortion at R_2	=	0.4335825594216250
WTE at R_1 - WTE at R_2	=	4.33680868994202E-19 TAN(TE) per mm
WTE at R_1 + WTE at Rw	=	3.68628738645072E-18 TAN(TE) per mm
%RMS Distortion	=	0.288398295068423

'LÖFGREN B' EXAMPLE USING FORMULAS

Purpose: To calculate the overhang d required to minimise the RMS distortion in circumstances where the offset angle β is known and non-optimum.

Formulas derived by Löfgren, except for the null radii formulas which are the roots of the tracking error equation.

d	= L - SQRT (L^2 - Ra^2)
Ra^2	= 3 * R_1 * R_2 * (*p* * (R_1 + R_2) - R_1 * R_2) / ((R_1 + R_2)^2 - R_1 * R_2)
p	= L * SIN (β * PI() / 180)
M	= L - d
Rw	= Ra^2 / *p*
R_{01}	= L * SIN (β * PI() / 180) - SQRT ((L * SIN (β * PI() / 180))^2 - L^2 + M^2))
R_{02}	= L * SIN (β * PI() / 180) + SQRT ((L * SIN (β * PI() / 180))^2 - L^2 + M^2))
WTE R_1	= TAN (ASIN ((R_1^2 + 2 * L * d - d^2) / (2 * L * R_1)) - β * PI() / 180) / R_1
WTE Rw	= TAN (ASIN ((Rw^2 + 2 * L * d - d^2) / (2 * L * Rw)) - β * PI() / 180) / Rw
WTE R_2	= TAN (ASIN ((R_2^2 + 2 * L * d - d^2) / (2 * L * R_2)) - β * PI() / 180) / R_2

Inputs:

Tonearm length L, inner groove radius R_1, outer groove radius R_2, the fixed but non-optimum offset angle β.

For this example, the entered offset angle is 21 degrees.

Results:

Overhang d	=	15.6358075334037 mm
Radius Rw	=	84.5323952556517 mm
d/dR of WTE at Rw	=	2.95327799127296E-09
Radius Ra	=	87.0254289818798 mm
d/dR of WTE at Ra	=	-2.04963515246941E-18
Radius Ra^2	=	7573.42528948020 mm^2
Inner null radius R_{01}	=	68.3011697798725 mm
Outer null radius R_{02}	=	110.882804992778 mm
Sine of offset angle β	=	0.358367949545300 mm
Linear offset p	=	89.5919873863251 mm
Mounting distance M	=	234.364192466596 mm
WTE at R_1	=	0.000238072753598986 TAN(TE) per mm
WTE at Rw	=	-0.000127966203292388 TAN(TE) per mm
WTE at R_2	=	0.000276978132389107 TAN(TE) per mm
%Distortion at R_1	=	0.5062619486759970
%Distortion at Rw	=	0.2721202593077780
%Distortion at R_2	=	0.5889942755908250
WTE at R_1 - WTE at R_2	=	-3.89053787901210E-05 TAN(TE) per mm
WTE R_1 + WTE Rw	=	1.10106550306599E-04 TAN(TE) per mm
%RMS Distortion	=	0.269868857763495

‘PERFECT LÖFGREN B’ EXAMPLE USING MICROSOFT EXCEL SOLVER

Purpose: To calculate the overhang d required to minimise the RMS distortion in circumstances where offset angle β is known and non-optimum.

Excel Solver settings:

Inputs:	Tonearm length L, inner groove radius R_1, outer groove radius R_2, the fixed but non-optimum offset angle β.
Calculates:	The required overhang d
Objective cell:	RMS distortion
Equal to:	Minimum.
Changing cells:	Overhang, radius Rw
Constraints:	The derivative of the WTE function at Rw = 0
Solving method:	GRG Nonlinear
Derivatives:	Central
Precision:	1E-15
Convergence:	1E-15

Note: For this example, the offset angle β, an input, is set to 21 degrees.

Results:

Overhang d	=	15.6303803304351 mm
Radius Rw	=	84.4992866678076 mm
d/dR of WTE at Rw	=	5.35704971051446E-15 TAN(TE) per mm^2
Radius Ra	=	87.0108118336077 mm
d/dR of TE at Ra	=	0.0 TAN(TE) per mm
Radius Ra^2	=	7570.88137594349 mm^2
Inner null radius R_{01}	=	68.2415113286832 mm
Outer null radius R_{02}	=	110.9424634439670 mm
Sine of offset angle β	=	0.358367949545300
Linear offset p	=	89.5919873863251 mm
Mounting distance M	=	234.369619669565 mm
WTE at R_1	=	0.000236566449350794 TAN(TE) per mm
WTE at Rw	=	-0.000128726187182023 TAN(TE) per mm
WTE at R_2	=	0.000276717975580127 TAN(TE) per mm
%Distortion at R_1	=	0.5030587911854370
%Distortion at Rw	=	0.2737363658093110
%Distortion at R_2	=	0.5884410518762900
WTE at R_1 - WTE at R_2	=	-4.01515262293336E-05 TAN(TE) per mm
WTE at R_1 + WTE at Rw	=	1.07840262168771E-04E-04 TAN(TE) per mm
%RMS Distortion	=	0.269864759373795

'LÖFGREN C' EXAMPLE USING FORMULAS

Purpose: To calculate the required offset angle and overhang to minimise the RMS distortion.

New Formula derived by Vladan Jovanovic

Löfgren did not provide a formula-based solution for the 'Löfgren C' alignment, but it now exists, thanks to an article by Vladan Jovanovic [18]. The article provides a formula for the linear offset *p*, shown below. From $p = L \cdot \sin(\beta)$, the offset angle β may be readily determined. The solution was achieved using partial derivatives of the distortion integral. Löfgren partly solved this alignment by providing a formula for the distortion-minimising *overhang*, as is also used in the 'Löfgren B' formula solution to minimise the LMS/RMS distortion but did not provide a formula for the distortion-minimising *offset angle* to accompany it. That formula has been provided by Jovanovic. This formula-based solution is an historical first!

As noted above, the required offset angle β is calculated from the linear offset *p*. The value of *p* is substituted into Löfgren's EQN (48) to calculate Ra^2 (shown below), after which the value of Ra^2 is substituted into Löfgren's EQN (47) to calculate the overhang d (shown below).

The resulting alignment is an excellent approximation to the 'perfect Löfgren C' alignment shown on the following page. This is a great step forward in the history and development of tonearm alignment. Also refer to pages S9-22 and S9-23 for more details.

p = ((L^2 + R_1 * R_2) * (R_2 - R_1)^3 - SQRT ((R_2 - R_1)^6 * (L^2 + R_1 * R_2)^2 - (2 * L * R_1 * R_2)^2 * (2 * (R_1^2 + R_1 * R_2 + R_2^2) * LN (R_2 / R_1) -3 * (R_2^2 - R_1^2))^2)) / (4 * R_1 * R_2 * (R_1^2 + R_1 * R_2 + R_2^2) * LN (R_2 / R_1) - 6 * R_1 * R_2 * (R_2^2 - R_1^2))
Sin β = p / L
β = ASIN (p / L) * 180 / PI()
Ra^2 = (3 * R_1 * R_2 * (p * (R_1 + R_2) - R_1 * R_2)) / (R_1^2 + R_1 * R_2 + R_2^2)
d = L - SQRT (L^2 - Ra^2)
M = L - d
Rw = Ra^2 / p
R_{01} = p - SQRT (p^2 - Ra^2)
R_{02} = p + SQRT (p^2 - Ra^2)
WTE R_1 = TAN (ASIN ((R_1^2 + 2 * L * d - d^2) / (2 * L * R_1)) - β * PI() / 180) / R_1
WTE Rw = TAN (ASIN ((Rw^2 + 2 * L * d - d^2) / (2 * L * Rw)) - β * PI() / 180) / Rw
WTE R_2 = TAN (ASIN ((R_2^2 +2 * L * d - d^2) / (2 * L * R_2)) - β * PI() / 180) / R_2

Inputs:

Tonearm length L, inner groove radius R_1, outer groove radius R_2

Results

Offset angle β	=	21.8284270203502 degrees
Overhang d	=	16.7979879740325 mm
Radius Rw	=	87.3179178219088 mm
d/dR of WTE at Rw	=	3.60764061458152E-09 TAN(TE) per mm^2
Radius Ra	=	90.0934047921406 mm
d/dR of TE at Ra	=	0.0 TAN(TE) per mm
Radius Ra^2	=	8116.82158704051 mm^2
Inner null radius R_{01}	=	70.0616260332851 mm
Outer null radius R_{02}	=	115.852600725886 mm
Sine of offset angle β	=	0.37182845351834200
Linear offset p	=	92.9571133795855 mm
Mounting distance M	=	233.202012025967 mm
WTE at R_1	=	0.000321391769978008 TAN(TE) per mm
WTE at Rw	=	-0.000138814140929103 TAN(TE) per mm
WTE at R_2	=	0.000233497572360512 TAN(TE) per mm
%Distortion at R_1	=	0.6834399203512500
%Distortion at Rw	=	0.2951884095436080
%Distortion at R_2	=	0.4965328211957590
WTE at R_1 - WTE at R_2	=	8.78941976174958E-05 TAN(TE) per mm
WTE at R_1 + WTE at Rw	=	1.82577629048905E-04 TAN(TE) per mm
%RMS Distortion	=	0.258400764431864

'PERFECT LÖFGREN C' EXAMPLE USING MICROSOFT EXCEL SOLVER

Purpose: To calculate the required offset angle and overhang to minimise the RMS distortion.

Excel Solver settings:

Inputs:	Tonearm length L, inner groove radius R_1, outer groove radius R_2
Calculates:	The required offset angle and overhang
Objective cell:	RMS distortion
Equal to:	Minimum.
Changing cells:	Offset angle, overhang, radius Rw
Constraint:	The derivative of the WTE function at Rw = 0
Optional constraint:	WTE at R_2 = Q x WTE at R_1. Refer below.
Solving method:	GRG Nonlinear
Derivatives:	Central
Precision:	1E-14
Convergence:	1E-14

Results:

Offset angle β	=	21.8405154174314 degrees
Overhang d	=	16.8091336379488 mm
Radius Rw	=	87.3217370283733 mm
d/dR of WTE at Rw	=	1.31399479953731E-15 TAN(TE) per mm^2
Radius Ra	=	90.1222496209555 mm
d/dR of TE at Ra	=	0.0 TAN(TE) per mm
Radius Ra^2	=	8122.01987674182 mm^2
Inner null radius R_{01}	=	70.0254282336725 mm
Outer null radius R_{02}	=	115.9867219896020 mm
sin β	=	0.372024300446550
Linear offset p	=	93.0060751116374 mm
Mounting distance M	=	233.190866294669 mm
WTE at R_1	=	0.000320996548186016 TAN(TE) per mm
WTE at Rw	=	-0.000139768586496880 TAN(TE) per mm
WTE at R_2	=	0.000232584563187515 TAN(TE) per mm
%Distortion at R_1	=	0.6825994808152340
%Distortion at Rw	=	0.2972180389982330
%Distortion at R_2	=	0.4945913062760840
WTE at R_1 - WTE at R_2	=	8.84119849985015E-05 TAN(TE) per mm
WTE at R_1 + WTE at Rw	=	1.81227961689136E-04 TAN(TE) per mm
%RMS Distortion	=	0.258393438443246

Optional Alignment:

The 'Perfect Löf C' worksheet in the accompanying Excel Tonearm Alignment Workbook has an option to lower the distortion at the inner or outer groove areas, if required, to minimise the distortion in the region of a particular record track of interest to the listener.

This is achieved by adding a constraint to the Excel Solver, which is handled automatically, based on the value of the variable Q, which is entered on the worksheet. Refer to the information in the 'Perfect Löf C' worksheet in the Tonearm Alignment Workbook for full details.

'STEVENSON A' EXAMPLE USING FORMULAS

Purpose: To calculate the required offset angle and overhang **which places the inner null radius at the inner groove radius,** i.e., to set the WTE at the inner groove radius to zero, and to equalise the magnitudes of the WTE at Rw and R_2.

The inner null radius R_{01} is an input, and in this example is set to 60.325 mm, the record's inner groove radius. (Although not used, and for information only, the inner groove radius R_1 calculates below at 54.8… mm.)

In summary, according to Stevenson's alignment method, the recorded groove surface runs between the inner null radii R_{01} and the outer radius R_2, i.e., the record's inner groove radius of 60.325 mm is located at the position of the inner null radius of the WTE plot. Stevenson's inner groove radius (R_1) is an unused, theoretical radius only.

Formulas provided by Stevenson (a rearrangement of Löfgren's formulas):

Sin β	= Ra^2 / (L * Rw)
β	= DEGREES (ASIN (SIN (β)))
d	= L - SQRT (L^2 - Ra^2)
Ra^2	= (R_{01}^2 * R_2) / ((2 * SQRT (2) - 2) * R_{01} - (2 * SQRT (2) - 3) * R_2)
M	= L - d
Rw	= (2 * R_{01} * Ra^2) / (R_{01}^2 + Ra^2)
R_{02}	= L * SIN (β * PI() / 180) + SQRT ((L * SIN (β * PI() / 180) $)^2$ - L^2 + M^2))
WTE R_{01}	= TAN (ASIN ((R_{01}^2 + 2 * L * d - d^2) / (2 * L * R_{01})) - β * PI() / 180) / R_{01}
WTE Rw	= TAN (ASIN ((Rw^2 + 2 * L * d - d^2) / (2 * L * Rw)) - β * PI() / 180) / Rw
WTE R_2	= TAN (ASIN ((R_2^2 + 2 * L * d - d^2) / (2 * L * R_2)) - β * PI() / 180) / R_2

Inputs:

Tonearm length L, inner null radius R_{01}, outer groove radius R_2.
In this example, R_{01} is set to 60.325 mm and R_2 is set to 146.05 mm.

Results:

Offset angle β	=	20.8236114003234 degrees
Overhang d	=	14.5927506993955 mm
Radius Rw	=	79.7027620698773 mm
d/dR of WTE at Rw	=	1.01895610872089E-08 TAN(TE) per mm^2
Radius Ra	=	84.1630974758119 mm
d/dR of TE at Ra	=	-2.18170944097205E-18 TAN(TE) per mm
Radius Ra^2	=	7083.42697672301 mm^2
Inner groove radius R_1	=	54.8057428323774 mm
Outer null radius R_{02}	=	117.421085399470 mm
Sine of offset angle β	=	0.355492170798940
Linear offset p	=	88.8730426997349 mm
Mounting distance M	=	235.407249300605 mm
WTE at R_{01}	=	4.60100714722402E-18 TAN(TE) per mm
WTE at Rw	=	-0.000245328447062990 TAN(TE) per mm
WTE at R_2	=	0.000248063587720868 TAN(TE) per mm
%Distortion at R_1	=	0.0000000000000098
%Distortion at Rw	=	0.5216911880851800
%Distortion at R_2	=	0.5275074674301600
WTE at Rw + WTE at R_2	=	2.73514065787789E-06 TAN(TE) per mm
%RMS Distortion	=	0.349943775980539

'PERFECT STEVENSON A' EXAMPLE USING MICROSOFT EXCEL SOLVER

Purpose: To calculate the required offset angle and overhang which places the inner null radius at the inner groove radius, i.e., to set the WTE at the inner groove radius to zero, and to equalise the magnitudes of the WTE at Rw and R_2.

The inner null radius R_{01} is an input, and in this example is set to 60.325 mm, the record's inner groove radius. (Although not used, and for information only, the inner groove radius R_1 calculates below at 54.8… mm.)

In summary, according to Stevenson's alignment method, the recorded groove surface runs between the inner null radii R_{01} and the outer radius R_2, i.e., the record's inner groove radius of 60.325 mm is located at the position of the inner null radius of the WTE plot. Stevenson's inner groove radius (R_1) is an unused, theoretical radius only.

Excel Solver settings:

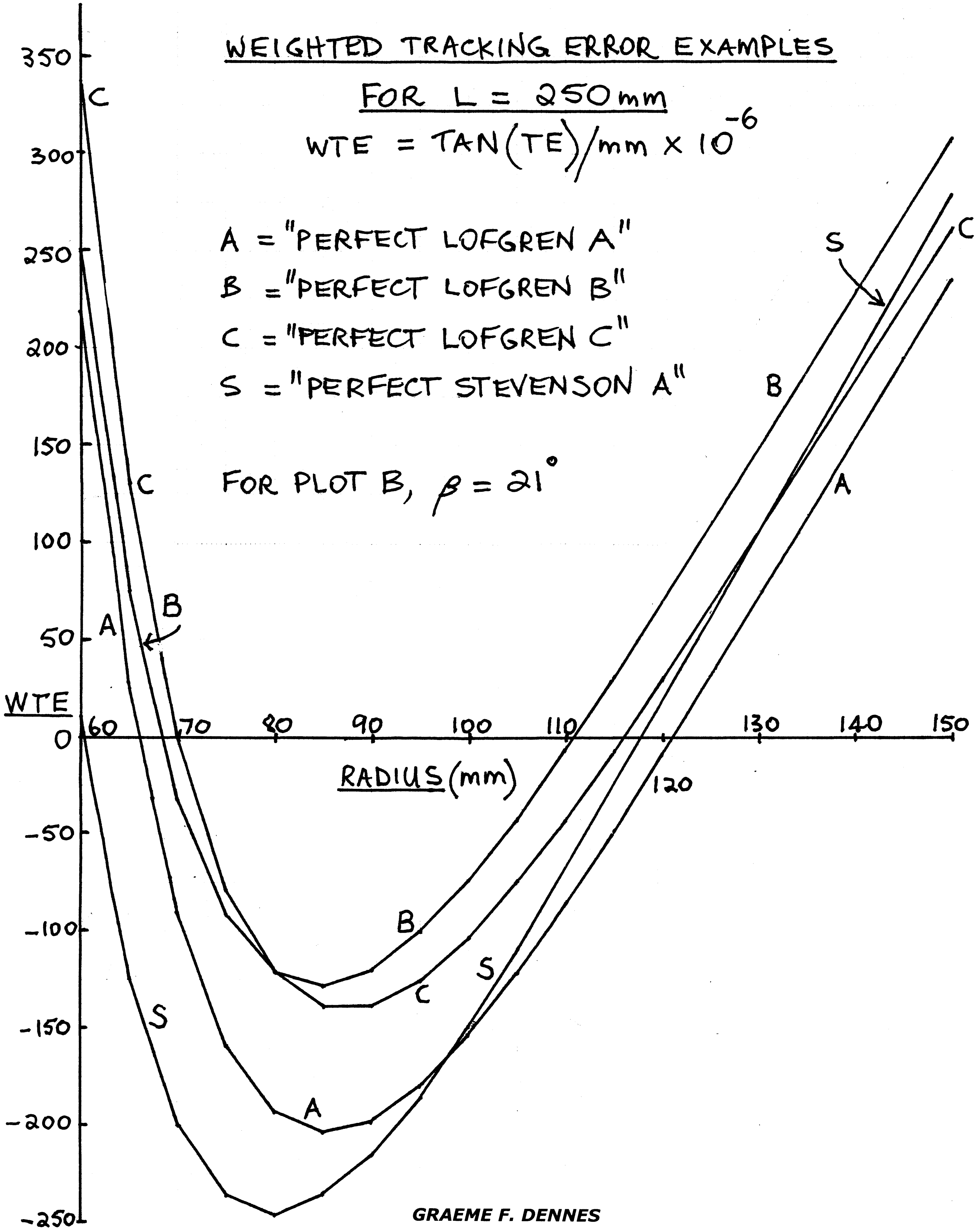
Inputs:	Tonearm length L, inner null radius R_{01}, outer groove radius R_2
Calculates:	The required offset angle and overhang
Objective cell:	WTE R_{01}
Equal to value of:	0
Changing cells:	Offset angle β, overhang d, radius Rw
Constraints:	1. The derivative of the WTE function at Rw = 0
	2. WTE R_W + WTE R_2 = 0
Solving method:	GRG Nonlinear
Derivatives:	Central
Precision:	1E-15
Convergence:	1E-15

Inputs:

In this example, R_{01} is set to 60.325 mm and R_2 is set to 146.05 mm.

Results:

Offset angle β	=	20.8456722718271 degrees
Overhang d	=	14.6158059369011 mm
Radius Rw	=	79.7303706887492 mm
d/dR of WTE at Rw	=	4.74338450462408E-20 TAN(TE) per mm^2
Radius Ra	=	84.2275559734782 mm
d/dR of TE at Ra	=	0.0 TAN(TE) per mm
Radius Ra^2	=	7094.28118526540 mm^2
Radius R_1 (for information only)	=	54.8449085472690 mm
Outer null radius R_{02}	=	117.6010142605120 mm
Sine of offset angle β	=	0.355852028521024
Linear offset p	=	88.9630071302561 mm
Mounting distance M	=	235.384194063099 mm
WTE at R_{01}	=	9.20201429444804E-19 TAN(TE) per mm
WTE at Rw	=	-0.000246530078856156 TAN(TE) per mm
WTE at R_2	=	0.000246530078856157 TAN(TE) per mm
%Distortion at R_{01}	=	0.0000000000000020
%Distortion at Rw	=	0.5242464592953590
%Distortion at R_2	=	0.5242464592953600
WTE at Rw + WTE at R_2	=	0.0 TAN(TE) per mm
%RMS Distortion	=	0.350751816638718



GRAEME F. DENNES

WRITER'S RESPONSES ON PUBLIC FORUMS

Post 1:
As a background note re the 'Löfgren A' alignment and the null points. Null points are calculated from the inner and outer groove radii selected for alignment optimisation purposes. They are outputs only. They are not the drivers of the optimisation philosophy or arrangement, but rather the consequences of it. Of course, they may be specified for use in an alignment procedure or for when a particular alignment tool is being used, but their specification automatically pre-determines the values of the inner and outer groove radii.

As a background note re the 'Stevenson A' alignment. Stevenson's goal was to make zero the weighted tracking error (WTE) at the specified inner groove radius. However, under the 'Stevenson A' alignment, the WTE (and the distortion) occurring over about 75 percent of the record playing time is greater than that which occurs under the 'Löfgren A' alignment for the same conditions (ie arm length and groove radii). Further, the 'Stevenson A' alignment is only significantly better than the 'Löfgren A' alignment during the last 3-4 mm of the record playing surface (usually less than one minute of playing time). It is primarily for these reasons that the 'Stevenson A' alignment has never become broadly accepted.

Practically, this leaves us to choose between the 'Löfgren A' and 'Löfgren C' alignments, although the 'Löfgren A' alignment still continues to this day to be the most widely utilised alignment strategy, given that it minimises tracking distortion to the smallest extent possible.

Post 2:
To add a note to my post above:

For the 'Löfgren A' solution, the null radii are dependent only on the selected inner (R_1) and outer (R_2) groove radii, and are given by the EQNs C1 and C2 on page S9-3 of my tonearm book. Baerwald was the first to explicitly state these equations, although I presume Löfgren knew of them because his a^2 term (Ra^2) at EQN E2 on page S9-4 of my tonearm book is the product of the null radii equations shown on page S9-3.

Löfgren and the later authors utilised minor mathematical approximations to simplify the analysis and derivation of their solutions. Note that the page S9-3 null radii equations are consistent with the 'Löfgren A' solution only.

Mathematically, the null radii are in fact the roots of the weighted tracking error (WTE) function, ie, are the radii at which the WTE function is zero. The WTE function is shown at EQN J1 on page S9-6, while the accurate null radii equations are shown at EQNs N1 and N2 on page S9-8. You'll note that the null radii are only dependent on arm length, offset angle and overhang, and not on the groove radii. Any variation in these three parameters will result in a change to the null radii values. Also, the null radii equations given on page S9-8 are correct for all values of arm length, offset angle and overhang, not just the optimum values determined by the 'Löfgren A' solution.

Thus, the null radii calculated from the page S9-3 equations will be close to, but slightly different from, the theoretically correct null radii calculated from page S9-8 equations.

In summary, to calculate the null radii: When using the 'Löfgren A' alignment, the null radii equations on page S9-3 are practical and consistent, although the page S9-8 equations will result in a more accurate solution. If using the 'Stevenson A' alignment, use either EQN C1 on page S9-22 (noting that R_{01} is already specified as an input), or EQNS N1 and N2 on page S9-8. If using 'Löfgren C' or *any* other alignment, the page S9-8 equations *must* be used.

Post 3:
Further to the issue regarding differences between the various alignments and their impacts on distortion over the playing surface of a record, may I refer you to page S1-28 of my tonearm book. On that page is a graph which shows the weighted tracking error (WTE) across the groove radii of 60 to 150 mm for four alignment strategies. Tracking distortion is proportional to the WTE, as was shown by Löfgren.

Now, a closer look at the four plots on that graph may help explain the effects of the different alignment strategies, ie, different offset angle-overhang pairs. After all, the only differences between the plots are due to the differences in their offset angle and overhang values. Changes to those values will result in changes to the shape of the WTE plot. The null radii are located at those groove radii where the WTE is zero. Areas on the plots with the higher WTE values – positive and negative WTE values are treated equally for distortion purposes - will produce higher distortion at those groove radii.

In the 'Löfgren A' alignment, the offset angle and overhang are adjusted to minimise the magnitudes of the WTE (and tracking distortion) by equalising and minimising the three peaks across the playing surface.

In the 'Löfgren B' alignment, for a fixed, non-optimum offset angle, the overhang is adjusted to obtain minimum LMS distortion.

In the 'Löfgren C' alignment, the offset angle and overhang are adjusted to obtain minimum LMS distortion, which has the effect of reducing the WTE (and tracking distortion) between the null radii but increases it at the inner and outer groove areas.

In the 'Stevenson A' alignment, the WTE at the inner radius is set to zero by the required offset angle and overhang values, but the WTE is increased for around 75 percent of the record playing surface when compared to the 'Löfgren A' alignment.

We note that the WTE for the four alignments over the outer one third of the playing surface (say from the outer grooves to 115 mm radius) are somewhat similar. However, the WTE for the plots over the inner two-thirds of the playing surface are quite different, resulting in different WTE and distortion characteristics for that area of the record.

In summary, the graphs on page S1-28 are four examples of different alignments, and the equations used to obtain their offset angle and overhang values are shown in my tonearm book, courtesy of their authors: Löfgren in 1938 and Stevenson in 1966.

Graeme Dennes

QUICK AND NOVEL 'LÖFGREN A' ALIGNMENT CALCULATIONS

The following equations provide a quick and novel (but reasonably accurate) method of calculating the optimum alignment parameters for the 'Löfgren A' alignment, certainly good enough for practical alignment calculations. Using these equations, the optimum offset angle, the optimum overhang, and the resulting maximum |WTE| (at the three WTE peaks) may be easily calculated. The only input required is the arm length in mm. The equations utilise three numbers which remain essentially constant over the range of alignment values likely to be encountered in practice.

Also included for comparison purposes are the results for the 'perfect Löfgren A' solution. You may be surprised at the accuracy of these equations!

The equations are based on an inner groove radius of 60.325 mm, and an outer groove radius of 146.05 mm.

Notation: L = arm length, β = optimum offset angle, d = optimum overhang, WTE = weighted tracking error.

1. Quick calculation of Offset Angle β:

   Let $L \cdot \sin\beta \approx 93.516$ mm (Löfgren's Linear Offset)

   then $\sin\beta = 93.516 / L$

   so $\beta = \arcsin(93.516 / L)$

   For L = 250 mm, $\sin\beta = 0.374064$: $\beta = 21.966$ degrees

   The 'perfect Löfgren A' result? $\beta = 21.963$ degrees

2. Quick Calculation of Overhang d:

   Let $2 \cdot L \cdot d - d^2 \approx 7987\ \text{mm}^2$ (Löfgren's Ra^2 term)

   then d = L – SQRT(L^2 - 7987) mm

   For L = 250 mm: d = 16.5180 mm

   The 'perfect Lofgren A' result? d = 16.5198 mm

3. Quick Calculation of Maximum |WTE| value at the three peaks:

   Let $L \cdot \cos\beta$ * max. |WTE| ≈ 2.709 degrees

   then max. |WTE| = 2.709 / ($L \cdot \cos\beta$) degrees per mm

   For L = 250 mm and $\beta = 21.966$ degrees:

   max. |WTE| = 0.01168 degrees per mm

   The 'perfect Löfgren A' result? = 0.01168 degrees per mm!

CALCULATING TRACKING DISTORTION

Tracking Distortion

Löfgren showed that at lower distortion levels, tracking distortion is principally second harmonic in nature. He developed an expression for the distortion ε and shows it to be the product of two factors in his EQN (22). This is an historically significant expression, as Löfgren was the first to show the relationship between the four variables involved in the generation of tracking distortion. A summary of Löfgren's derivation of this expression is included in Section S10 of this book.

Löfgren's EQN (22) is: $\varepsilon \approx \frac{V}{\Omega} \cdot \frac{ABS(\delta)}{R}$

where ε is the distortion factor, V is the maximum peak recorded velocity, Ω is the turntable speed, δ is the tracking error in radians and *R* is the radius. The term δ (radians) / R is the weighted tracking error and the term δ (radians) is the tracking error.

In the releases of this book prior to the 2021 release, the distortion term ε above was used as the basis for distortion calculations. In fact, the entire book, including all formulas and derivations, was centred on Löfgren's EQN (22) above.

High-accuracy Distortion Calculations

In the lead-up to his EQN (22) above, Löfgren developed a high-accuracy, no-approximation distortion term ε at EQN (14a) which uses the term TAN(δ) instead of δ, as follows:

EQN (14a): $\varepsilon = \mathrm{TAN}(\alpha) \cdot \mathrm{ABS}(\mathrm{TAN}(\delta))$

Then from Löfgren's EQNs (21):

EQN (21): $\mathrm{TAN}(\alpha) = \frac{\mathrm{V}}{\Omega\,\mathrm{R}}$

Substituting EQN (21) into EQN (14a), the high-accuracy distortion formula is:

$\varepsilon = \frac{V}{\Omega} \cdot \frac{\mathrm{ABS}(\mathrm{TAN}(\delta))}{\mathrm{R}}$, referred to here as **EQN (21A)**.

The term TAN(δ) / R is the high-accuracy WTE term and TAN(δ) is the high-accuracy TE term involved in the generation of distortion. This shows that the tracking distortion ε is proportional to the maximum peak recorded velocity *V* and the absolute value of the trigonometric tangent of the tracking error δ, and is inversely proportional to the angular velocity of the record Ω and the radius R. As Löfgren notes, the first factor in EQNs (21A)/(22) is independent of the position of the needle on the record, while the second factor changes continually during play. The high accuracy distortion formula is also shown as the last formula on page S10-9 in this book.

As was stated by Löfgren, EQN (22) resulted from using δ (radians) as an acceptable approximation for TAN(δ) in EQN (14a) when δ is less than 10 degrees.

(The remaining sections of this book are generally based on Löfgren's EQN (22) for historical purposes, for consistency with Löfgren's article and with much of the mathematical material presented later in this book.)

Note: Throughout this book, all references to EQN (21A) above may also generally apply to Löfgren's EQN (22), except for when distortion figures based on EQN (21A) are being calculated.

We also must ensure the units are consistent. For example, if radius R is in mm, then maximum peak recorded velocity *V* must be in mm per second, and if δ is in radians, then Ω has to be in radians per second.

For a turntable speed of 33 1/3 RPM, the angular velocity Ω of the record equals:

Ω = 100 / 3 / 60 * 2 * PI
= 10 * PI / 9 radians per second.

For the LP record with a maximum peak recorded velocity *V* of 100 mm per second and tracking error δ in radians (Excel's TAN function requires angle arguments in radians), then from EQN (21A):

Distortion ε = 100 / (10 * PI / 9) * ABS(TAN(δ)) / radius R

= 90 / PI * ABS(TAN(δ)) / R **EQN (21B)**

= 90 / PI * ABS(WTE) **EQN (21C)**

For example, with a tracking error of 2 degrees at a radius of 130 mm, then from EQN (21B), the tracking distortion ε = 90 / PI * ABS(TAN(2 * PI / 180)) / 130 = 0.00769543, ie, the second harmonic level is 0.00769543 of the fundamental or 0.769543%.

In Löfgren's article, the distortion term ε is also suitable to act as a distortion factor K for calculating LMS/RMS tracking distortion, i.e., from EQN (21C) above, let the distortion factor K = 90 / PI * ABS(WTE). More follows.

RIAA Correction Factor

An added factor which Löfgren did not include in the distortion calculations because there was no single standard at the time, and which needs to be included, is the effect on the harmonic distortion components by the frequency de-emphasis characteristic of the RIAA stage of the phono preamplifier. We will consider the distortion resulting when playing a recorded 1 KHz test tone.

For a fundamental frequency of 1 KHz, the standard (three time constant) RIAA playback response curve shows a response of 0.0889815809580368 dB. For its second harmonic at 2 KHz, it shows a response of −2.49955932706491 dB.

This means the RIAA response curve will attenuate the second harmonic by 2.58854090802295 dB below the level of the 1 KHz fundamental. To convert this attenuation figure in dB to a gain figure, the RIAA correction factor becomes:

10^(-2.58854090802295 / 20) or 0.742288880049004 times.

This means the 2 KHz second harmonic distortion component is attenuated (multiplied) by this figure through the action of the preamp's RIAA frequency de-emphasis circuitry.

This RIAA correction factor can be applied in three possible equivalent ways:

1. As a multiplier for K before K is squared, i.e.
   K = 90 / PI * ABS(WTE) * 0.742288880049004.
2. As a squared multiplier for the mean distortion, i.e.
   mean = mean * $(0.742288880049004)^2$
3. As a multiplier for the RMS distortion, i.e.
   RMS = RMS * 0.742288880049004

Stevenson's Approximation to the RIAA Correction Factor

In accordance with the RIAA playback response curve, over the frequency range from 20Hz to 20KHz, which is approximately 10 octaves, the gain changes by approximately -40dB, or an average of -4dB per octave, as Stevenson states. (This is a very crude approximation to the true RIAA curve response shown on the previous page.) For the harmonic distortion components, this has the effect of attenuating the second harmonic by 4dB with respect to the fundamental, which means the distortion is lowered by this amount.

This is a gain change of $10^{-4/20}$, so we must allow for this by multiplying the distortion term ε by $10^{-4/20}$.

Thus, distortion $\varepsilon = 0.5 * 10^{-4/20} *$ ABS(WTE)

In summary, the constant $0.5 * 10^{-4/20}$ converts the ABS(WTE) figure to a second harmonic distortion figure.

As a brief aside, we can convert the distortion constant $\varepsilon = 0.5 * 10^{-4/20}$, which is approximately 0.3155, to the constant 1.76 used by Stevenson on page 215 of his article. Stevenson uses 100 mm/sec RMS (not peak) recorded velocity, so we must multiply the distortion constant by the square root of 2. He also uses radius values in *inches* (not mm), so we must divide the distortion constant by 25.4. Stevenson also calculated percentage distortion, so we must multiply the distortion by 100. Thus, the distortion constant 0.3155 becomes 0.3155 * SQRT(2) * 100 / 25.4 = 1.7566, or 1.76, per Stevenson's article. Thus, from Stevenson's article, the maximum percentage distortion $\varepsilon = 1.76 *$ WTE, where WTE is in degrees per inch.

LMS/RMS Distortion

Löfgren based his 'Löfgren B' alignment strategy and solution (as well as his 'Löfgren C' alignment strategy but without providing a solution) on minimising the LMS (Least Mean Squares) distortion resulting from tracking error. We can achieve the same outcome by minimising the RMS distortion, which will be done here.

The RMS value of a *varying* quantity is a single figure which represents a *statistical* measure of the *magnitude* of some varying quantity over some parameter range. In this application, the varying quantity is the distortion ε occurring over the range of groove radii of a record. The RMS distortion is a single figure representing *all these distortion levels*. The %RMS distortion is simply the RMS distortion multiplied by 100.

Procedure to Determine RMS Distortion

To determine the RMS distortion, we perform two fundamental steps:

1. Define a distortion function or distortion factor which expresses the level of distortion being produced at any instant.

2. Apply the necessary mathematical procedure to the distortion factor to calculate the RMS distortion.

EQNs (21A) and (22) are suitable distortion factors as they indicate the level of second harmonic distortion caused by tracking error.

The calculation of the tracking angle is based on the trigonometric inverse sine function. (The original WTE is calculated using EQN J on page S9-6.)

The RIAA correction still needs to be applied to EQNs (21A)/(22) as discussed.

In summary, the underlying method to calculate the RMS distortion includes the *integration* of the *square* of some distortion function K. In this case, the function is EQN (21A), where the ABS(WTE) part is given by ABS(TAN(δ)) / R. EQN (22) is less suitable because it uses the approximation δ (radians) for TAN(δ).

The Problem?
EQN (21A) is a suitable distortion term to use for calculating LMS or RMS distortion, but we cannot make use of it directly as there is no known closed form integral of EQN (21A)/(22) because of the inverse sine function used in the tracking angle calculation in the WTE part. Refer to page S1-47 for further details.

The Solution
In general, there are two methods available to overcome this as follows:

1. Use an approximation for the WTE part of EQNs (21A)/(22) which does not use the inverse sine function, and integrate the resulting function. This results in the ***integration of an approximation*** to EQNs (21A)/EQN (22), or

2. Apply a numerical integration procedure to EQNs (21A)/(22). One such procedure is the Composite Trapezoidal Rule. Another is the Tanh-Sinh method. This results in the ***approximate integration*** of EQNs (21A)/(22).

Note that because the first method is based on finding an approximation to the WTE part of EQNs (21A)/(22), and the second method is based on approximating the integral of EQNs (21A)/(22) using a numerical integration method, some inaccuracy will result with each method.

We'll consider each method in turn, then finally, we'll compare the results.

Method 1: Approximating the WTE Function
Löfgren realised that the inverse sine function and its argument in EQN (22) were causing difficulties on two fronts. Firstly, in the development of his 'Löfgren A' solution because it added much complexity to the analysis, and secondly, in the development of his 'Löfgren B' solution because of its intractable form, i.e. it could not be mathematically integrated as required for calculating the RMS distortion, on which the 'Löfgren B' alignment is based.

As a way around the "difficult" inverse sine function, Löfgren developed his EQN (33) as an excellent approximation to the WTE part of EQN (22). EQN (33) does not use the inverse sine function, so Löfgren was able to use EQN (33) in the derivation of several of his alignment equations for the 'Löfgren A' alignment.

However, for the development of the RMS distortion calculation for the 'Löfgren B' solution, EQN (33) still has integration complexities because of the remaining sine and cosine functions, so Löfgren derived the WTE part of his EQN (45) from his EQN (33). So now, EQN (45) is still equivalent to EQN (22), but the WTE part of EQN (45) does not use *any* trigonometric functions, so may be readily integrated for use in the RMS distortion calculations for his 'Löfgren B' solution, which quickly followed.

The complete derivation of EQNs (22) and (33) is included in Section S10, and the derivation of his EQN (45) is included in Section S11.

a. Mean and RMS Distortion Equations

Löfgren provides an equation for the mean distortion at EQN (46), which is based on the integral of the square of EQN (45). The RMS distortion value is then obtained by taking the square root of the mean distortion. EQN (46) is also derived in Section S11. Please note: The equation for the WTE part of EQN (45), ie, the second part or the δ / R part, is based on calculating the WTE in units of *radians per mm*, and so the turntable angular velocity Ω *must be specified in units of radians per second* for consistency.

Two examples of the RMS distortion calculation based on Löfgren's EQN (46) are shown in Sections S9 and S11. *Note that neither of these includes the RIAA correction.*

b. Applying the RIAA Correction

The RIAA correction is needed to be included in distortion calculations because of its effects on the reproduced sound.

We consider that the previously-discussed RIAA correction term, 0.742288880049004, is a part of the EQNs (21A)/(22) or EQN (45), being that it remains a constant during playback. To summarise, we have the choice of applying it in our distortion calculations in either of three ways, as follows:

1. We can apply it to the distortion factor equation, such as EQNs (21A)/(22) or EQN (45), *before* the distortion factor is squared and integrated. It's simply used as a multiplier for the fixed part, so the RIAA-corrected fixed part of both those equations becomes:

$$\frac{V}{\Omega} * 0.742288880049004$$

2. We can also apply the RIAA correction *after* the mean distortion has been calculated, such as with Löfgren's EQN (46). In this case, the mean distortion figure is multiplied by the *square* of the RIAA correction, so the RIAA-corrected mean distortion is:

$$\text{Mean distortion} \quad = \quad \text{Mean distortion} * (0.742288880049004)^2$$

3. We can apply it to the RMS distortion as a multiplier, as

$$\text{RMS distortion} = \text{RMS distortion} * 0.742288880049004$$

It doesn't matter which of the three ways we choose to apply the RIAA correction, as they are mathematically equivalent. However, the third method will be used here.

The mean distortion is calculated by dividing the integral of the squared distortion function by the interval of integration, and the RMS distortion is calculated by taking the square root of the mean distortion.

To summarise, this concludes the method of approximating the WTE part of the distortion factor, EQNs (21A)/(22), so as to remove the inverse sine function. This has been done by substituting EQN (33) for the WTE part. We can also use EQN (45) as a substitute for EQNs (21A)/(22), in which case, no trigonometric functions are used. Note that the WTE part of EQN (45) is equivalent to EQN (33), as shown elsewhere in this book. We've also shown how the RIAA correction is applied. An example follows.

c. RMS Distortion

We'll now go through an example using Löfgren's EQN (45), which is the distortion factor, and EQN (46), which is the mean distortion. EQN (45) is based on EQN (33), but it uses no trigonometric functions as noted. We'll also show how the RIAA correction may be applied to either the distortion factor, EQN (45), before the integration is done, or to the mean distortion, EQN (46), after the integration is done.

We'll look at the equations first, then apply the test data and determine the RMS distortion.

From Löfgren's EQN (22), which calculates the tracking distortion, we can use it as a distortion factor K, i.e.

$$\text{Distortion factor K} = \frac{V}{\Omega} \cdot \frac{S}{R}$$

$$= \frac{V}{\Omega} \cdot WTE$$

From Löfgren's EQN (45):

$$\text{Distortion factor K} = \frac{V}{\Omega} \cdot \frac{1}{(L^2-\rho^2)^{\frac{1}{2}}} \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} \right]$$

We now apply the RIAA correction, i.e.

$$K = \frac{V}{\Omega} \quad * \quad 0.742288880049004 \quad * \quad \frac{1}{(L^2-\rho^2)^{\frac{1}{2}}} \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} \right]$$

The distortion factor K is now squared, i.e.

$$K^2 = \frac{V^2}{\Omega^2} \quad * \quad (\ 0.742288880049004\ )^2 \quad *$$

$$\frac{1}{(L^2-\rho^2)} \left[\frac{1}{4} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} + \frac{\rho^2}{R^2} - \frac{\rho Ra^2}{R^3} + \frac{Ra^4}{4R^4} \right]$$

This equation is now integrated with respect to the radius R.

After the integration, we obtain for the mean distortion:

$$\text{Mean} = \frac{V^2}{\Omega^2} * (0.742288880049004)^2 * \frac{1}{L^2 - \rho^2} * \frac{1}{R_2 - R_1} *$$

$$\left| \frac{R}{4} - \rho \ln R - \frac{Ra^2}{2R} - \frac{\rho^2}{R} + \frac{\rho Ra^2}{2R^2} - \frac{Ra^4}{12R^3} \right|_{R_1}^{R_2}$$

After the limits of integration are applied, we have for the mean distortion:

$$\text{Mean} = \frac{V^2}{\Omega^2} * (0.742288880049004)^2 * \frac{1}{L^2 - \rho^2} * \frac{1}{R_2 - R_1} *$$

$$\left[\frac{R_2 - R_1}{4} - \rho \ln \frac{R_2}{R_1} + \frac{Ra^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \rho^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) - \rho \frac{Ra^2}{2}\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) + \frac{Ra^4}{12}\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) \right]$$

For calculations, separate the above equation into its seven parts as follows:

Mean = Term 1 * (Term 2 - Term 3 + Term 4 + Term 5 – Term 6 + Term 7)

The equation above is identical to Löfgren's EQN (46), except for the application of the RIAA correction. Note that Ra^4 is equivalent to $(Ra^2)^2$.

It also confirms that the RIAA correction may be applied to the distortion factor before the integration, or to the mean distortion as a squared term.

d. RMS Distortion Example

We'll use the mean distortion equation above, which is Löfgren's EQN (46), and the following test data to calculate the mean distortion and the RMS distortion using Microsoft Excel, which uses 15 significant digits (SD) for its calculations. Note: Löfgren did not use the RIAA correction.

L = 250.00 mm
β = 22.0 degrees = 0.383972435438752 radians
d = 16.0 mm
R_1 = 60.325 mm
R_2 = 146.05 mm
Ω = 33 1/3 RPM = 3.49065850398866 radians per second
V = 100.0 mm per second maximum peak recorded velocity

As previously noted, Löfgren's mean distortion equation is comprised of seven terms - one outside the brackets and six within the brackets per the following. When the above parameters are substituted into the equation, the following values for the seven terms are obtained, along with the mean and RMS distortion results.

Mean distortion = 1.78182907563414 * 10^{-4} *

(21.4312500000000 - 82.8070138608386 + 37.6741923277897

+ 85.3374091296287 - 82.6451877596530 + 21.1603159396904)

= 0.0000268995210202244

RMS distortion = SQRT(Mean)

= SQRT(0.0000268995210202244)

= 0.00518647481631063

Apply the RIAA correction as a multiplier to the RMS distortion:

RMS distortion = RMS distortion * 0.742288880049004

= 0.00518647481631063 * 0.742288880049004

= 0.00384986258280158

%RMS distortion = 0.384986258280158%

Method 2: Approximating the Integral

We will now show how the RMS distortion may be calculated using the second technique above, where a numerical integration method is used to calculate the integral, from which the mean and RMS distortion figures are derived. We will use one of the common numerical integration techniques, namely, the **Composite Trapezoidal Rule** and implement it as a spreadsheet-based integrator in Microsoft Excel. Although not the most accurate numerical integration method – it manages to achieve eight correct digits out of the 15 digits used by Excel, it lends itself to spreadsheet-based calculations.

The spreadsheet calculations following are based on the *true* distortion factor at EQN (21A), EQN (21B) and EQN (21C) discussed earlier.

Some Excel formulas

Convert radians to degrees: degrees = radians * 180 / PI()

Convert degrees to radians: radians = degrees * PI() / 180

Tracking angle (radians) = ASIN $\left(\frac{R^2 + 2Ld - d^2}{2LR}\right)$

= ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R))

Tracking angle (degrees) = DEGREES(ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R)))

= ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R)) * 180 / PI()

Tracking error δ = tracking angle – offset angle (same angle units)

WTE = TAN(δ) / R

= TAN(tracking error) / R

= TAN(ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R)) – offset angle in radians) / R

The *true* distortion factor ε

= 90 / PI() * ABS (WTE)

= 90 / PI() * ABS (TAN (ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R)) – offset angle in radians)) / R

For mean and RMS distortion calculations, we use the square of the distortion factor, ε^2

= (90 / PI() * ABS (TAN (ASIN((R^2 + 2 * L * d – d^2) / (2 * L * R)) – offset angle in radians)) / R $)^2$

where R = radius in mm, L = arm length in mm, d = overhang in mm, offset angle in radians

Applying the RIAA Correction
As with Method 1, the RIAA correction for the RIAA frequency de-emphasis of the phono preamplifier may be applied in one of three ways. However, for this example, the correction will be applied to the RMS distortion as a multiplier:

RMS distortion = RMS distortion * 0.742288880049004

Spreadsheet Distortion Calculator
As already noted, a spreadsheet will be used to calculate the RMS distortion based on the true distortion factor and the true WTE formulas shown above to give greater accuracy of the results.

The general steps to calculate the RMS distortion using the Composite Trapezoidal Rule follow. We'll again use the parameters V = 100 mm/sec and Ω = 3.49065850398866 radians per second. (Remember, for consistency of units, because we use radius units in **mm**, we must use a maximum peak recorded velocity figure in **mm** per second, and because we use tracking error in **radians**, we must use a turntable speed in **radians** per second.)

1. From Löfgren's *true* distortion factor ε at EQN (21A), EQN (21B) and EQN (21C), we have V / Ω = 90 / PI and ABS(WTE) = ABS(TAN(δ)) / R .

2. Square the distortion factor, i.e. (90 / PI * ABS(WTE))^2 at each sampled radius.

3. Calculate the integral using the Composite Trapezoidal Rule, between the radius limits of 60.325 and 146.05 mm.

4. Mean distortion is the integral divided by the radius interval, (146.05 - 60.325).

5. RMS Distortion = Square root of the Mean distortion.

6. Apply the RIAA correction to the RMS distortion.

a. RMS Distortion Example

We'll perform the calculations required for the %RMS distortion using the 'PERFECT LOF C TRAP' Trapezoidal Rule spreadsheet with the following test data:

Let: arm length L = **250 mm**, offset angle β = **22.0 degrees**, overhang d = **16.0 mm**. We'll use radius increments in Excel of 0.005 mm, and step from 60.325 to 146.05 mm, for a total number of radius samples (rows) of 17,146, so number of panels is 17,145.

The spreadsheet is set up as follows:

Column 1 is a sequential **Line Number** listing of the row numbers used in the spreadsheet calculations. They run from 1 to 17146. The structure of the spreadsheet "table" (calculation cells) used in the calculations is shown below.

Column 2 lists the **Radius** values from 60.325 to 146.05 mm in 0.005 mm increments.

Column 3 is the **Tracking Angle in degrees** for each radius value, calculated by:

DEGREES(ASIN $\left(\frac{R^2 + 2Ld - d^2}{2LR}\right)$), where R = radius, L = arm length, d = overhang.

Column 4 is the **Tracking Error in degrees**, calculated by Tracking Angle in *Column 3* – Offset Angle (both in degrees).

Column 5 is the **WTE** in units of **TAN(TE)** per mm, calculated by TAN(TE) divided by the Radius. For example, *Column 5* = TAN(C*olumn 4* * PI / 180) / C*olumn 2*. (The TAN function uses angles in radians so we must first convert the TE figure in C*olumn 4* from degrees to radians by the conversion shown.)

Column 6 is the **Distortion Parameter (K)** (Tracking Distortion), calculated by 90 / PI * ABS(WTE) (Column 5), where 90 / PI = V / omega, and where V = 100 mm/sec maximum peak recorded velocity and omega = (100 / 3) / 60 * 2 * PI = record speed in radians/sec at 33 1/3 RPM.

Column 7 is the **Distortion Parameter Squared (K^2)** (Tracking Distortion squared), calculated by squaring *Column 6*.

Column 8 is the **Multiplier** used by the Composite Trapezoidal Rule. The Distortion Parameter Squared (*Column 7*) at the lowest radio (60.325 mm) and the highest radii (146.05 mm) are multiplied by 1, while the Distortion Parameter Squared (*Column 7*) for all intermediate radii are multiplied by 2, in accordance with the Composite Trapezoidal Rule.

Column 9 is the **Total**, the product of the **Distortion Parameter Squared** (*Column 7*) and the Multiplier (*Column 8*), in accordance with the Composite Trapezoidal Rule.

For the specified test data, the spreadsheet cell values should look like the following:

1	2	3	4	5	6	7	8	9
Line No.	**Radius (mm)**	**Tracking Angle (TA) (Degrees)**	**Tracking Error (TE) (Degrees)**	**WTE (TAN(TE) / mm)**	**Distortion Parameter (K)**	**Distortion Parameter Squared (K^2)**	**Multiplier**	**Total**
1	60.325	22.17227	0.172270	0.00004984	0.001427856	0.000002038771	1	0.000002038771
2	60.330	22.17157	0.171572	0.00004964	0.001421954	0.000002021953	2	0.000004043907
3	60.335	22.17087	0.170875	0.00004943	0.001416055	0.000002005213	2	0.000004010426
4	60.340	22.17018	0.170178	0.00004922	0.001410159	0.000001988550	2	0.000003977100
-	-	-	-	-	-	-	2	-
-	-	-	-	-	-	-	2	-
17143	146.035	23.46113	1.461126	0.00017466	0.005003741	0.000025037422	2	0.000050074845
17144	146.040	23.46152	1.461524	0.00017471	0.005004932	0.000025049348	2	0.000050098696
17145	146.045	23.46192	1.461921	0.00017475	0.005006124	0.000025061276	2	0.000050122552
17146	146.050	23.46232	1.462319	0.00017479	0.005007315	0.000025073207	1	0.000025073207

In accordance with the Composite Trapezoidal Rule:

Integral = *Sum of the Totals * Radius Interval / (2 * Number of Radius Panels)*

where: *Sum of the Totals* = sum of *Column 9* = 0.916188970242501

Radius Interval = 146.05 – 60.325 = 85.725 mm

Number of Radius Panels = Column 1 line number of the outer radius (146.05) minus the *Column 1 line number* of the inner radius (60.325)

= 17146 – 1
= 17145 panels

So the Integral = 0.916188970242501 * 85.725 / (2 * 17145)

= 0.00229047242560625

Mean Distortion (method 1) = *Integral / Radius Interval*

= 0.00229047242560625 / 85.725

=0.0000267188384439341

Mean Distortion (method 2) = *Sum of Totals / (2 * Number of Radius Panels)*

= 0.916188970242501 / (2 * 17145)

= 0.0000267188384439341

The mean distortion may be calculated by either of the above methods.

Then, RMS distortion = SQRT(Mean)

= SQRT(0.0000267188384439341)

= 0.00516902683722324

Now apply the RIAA correction:

RMS distortion = RMS distortion * 0.742288880049004

= 0.00516902683722324 * 0.742288880049004

= 0.00383691114194569

%RMS Distortion = RMS Distortion * 100

= 0.00383691114194569 * 100

= 0.383691114194569%

The above spreadsheet calculations and distortion figures are provided as a reference for an alignment based on an offset angle of 22.0 degrees and an overhang of 16.0 mm.

On the 'PERFECT LOF C TRAP' worksheet, we can also run the Solver tool to achieve the lowest possible distortion for the Löfgren C alignment. On the author's computer, after running the Solver to achieve the minimum distortion, the %RMS distortion figure returned was 0.258393445987143%, the offset angle was calculated at 21.8405154163698 degrees and the overhang was calculated at 16.8091336920305 mm.

b. Background to using the Composite Trapezoidal Rule integration method to calculate Mean distortion and RMS distortion

Integral = sum_of_totals * radius_interval / (2 * number_of_radius_panels)
= sum_of_totals * radius_interval / (2 * radius_interval / panel_width)
= sum_of_totals * (146.05 - 60.325) / (2 * (146.05 - 60.325) / 0.005)
= sum_of_totals * 85.725 / (2 * 85.725 * 200)
= sum_of_totals / 400

Mean distortion = sum_of_totals / (2 * number_of_radius_panels)
= sum_of_totals / (2 * radius_interval / panel_width)
= sum_of_totals / (2 * (146.05 - 60.325) / 0.005)
= sum_of_totals / (2 * 85.725 / 0.005)
= sum_of_totals / (2 * 85.725 * 200)
= sum_of_totals / 34290

So **Mean distortion** = integral / radius_interval
= sum_of_totals / 400 / radius_interval
= sum_of_totals / 400 / 85.725
= sum_of_totals / 34290

RMS distortion = SQRT (**Mean** distortion)

Apply the RIAA correction:

RMS distortion = RMS distortion * 0.742288880049004

%RMS distortion = RMS distortion * 100

RMS DISTORTION CALCULATION USING WEB-BASED INTEGRATOR

The web site https://www.integral-calculator.com' provides a free, online high-accuracy numerical integration program based on the Risch Algorithm.

For this RMS distortion calculation exercise, the parameters used will be the same as those used for Löfgren's EQN (46) integration calculator and for the Composite Trapezoidal Rule spreadsheet integration calculator, ie:

> Arm length = 250 mm; Offset angle = 22 degrees; Overhang = 16 mm;
> Inner radius = 60.325 mm, Outer radius = 146.05 mm.
> This sets 2Ld - d^2 = 7744, and 2L = 500 for use in the distortion expression.

Based on the previously discussed tracking distortion equation shown at EQN (21A), EQN (21B) and EQN (21C), and applying the above parameters, the function to be integrated is the **square** of the distortion function, which consists of two parts: the WTE part and the 90/PI part. The RIAA correction will be applied afterwards as a multiplier for the RMS distortion.

In Excel's terminology, the distortion expression to be integrated with respect to radius R is:

(TAN(ASIN((R^2+7744)/(500*R))-22*PI()/180)/R*90/PI())^2

where TAN is the tangent function, ASIN is the inverse sine function, R is the radius and PI() is the constant π.

The unit of angle used internally by both Excel and the web program is radians, so the offset angle (22 degrees) is entered in degrees and then converted to radians by multiplying the degrees by PI()/180 as shown in the formula, and the tracking angle and tracking error are calculated in radians. We then divide the TAN(tracking error in radians) result by the radius R in mm to give the WTE in TAN(radians) per mm. We then multiply the WTE by 90/PI to convert it to TAN(degrees) per mm, and finally, the formula is squared, all as shown in the expression above.

To use the web-based integration calculator:

1. Identify the function to be integrated, which is based on the function shown above, except that for the web program, all variables are to be in lower case. The integrating variable (the radius R) is required to be entered as 'r', the constant π is entered as 'pi', the TAN function is entered as 'tan' and the inverse sine function is entered as 'asin'.

   On the web page, the formula to be entered into the integrator's f(x) field is:

   (tan(asin((r^2+7744)/(500*r))-22*pi/180)/r*90/pi)^2

2. In the panel on the right, click the Options tab.

3. Click the Variable of integration box and select 'r'.

4. In the Upper bound (to) box, enter 146.05.

5. In the Lower bound (from) box, enter 60.325.

6. Check the box, Integrate numerically only.

7. Click the Go button. After a few seconds, the web program returns the following result in the Definite Integral panel lower down, which may be copied and pasted into Excel:

   Integral = 0.00229047241361082

   The known true result for the integral to 30 significant digits is:

   Integral = 0.00229047241361082841336377294871

   Note that the integral result is exact to all 15 digits.

8. The mean distortion is calculated by dividing the integral by the interval of integration, i.e., (146.05 - 60.325) = 85.725, as before, so

   Mean distortion = 0.00229047241361082 / 85.725

   = 0.0000267188383040049

9. The RMS distortion is calculated by taking the square root of the Mean, i.e.,

   RMS distortion = SQRT(Mean distortion)

   = SQRT(0.0000267188383040049)

   = 0.00516902682368789

10. The RIAA correction term is applied to the RMS distortion as follows:

   RMS distortion = RMS distortion * 0.742288880049004

   = 0.00516902682368789 * 0.742288880049004

   = 0.00383691113189854

11. %RMS distortion = RMS distortion * 100

   = 0.00383691113189854 * 100

   = 0.383691113189854%

Ranking of the %RMS Distortion Results for the Three Calculation Methods

1: Web integrator: %RMS Distortion = **0.383691113189854%**

2: Composite Trapezoidal Rule: %RMS Distortion = **0.38369111**4194569%

3: Löfgren's EQN (46): %RMS Distortion = **0. 38**4986258280158%

Discussion of the RMS Distortion Results in Order of Accuracy

The above results for the three RMS distortion calculations are ranked in order of accuracy, with the web integrator result accepted as being mathematically correct to the digits shown. It is used as the reference.

The two most accurate methods are based on the numerical integration of the true WTE function, ie, *with* the inverse sine function included, while the lowest ranking result is based on Löfgren's EQN (46), which utilises an approximation for the "difficult" arcsin function. All three calculation methods include the RIAA correction.

1: The web integrator program should provide the most accurate result of the three methods because of the high precision calculations used. All three results are shown to 15 SD for ease of comparison. This program has provided the perfect result for our purposes here, and is ranked number one for accuracy.

2: The second-best result is given by the Excel-based Composite Trapezoidal Rule integrator with radius increment of 0.005 mm and using 17,146 radius samples, where the result matched the first 8 digits of the web integrator result. This makes this method around half as accurate as the web integrator.

3: The result from Löfgren's EQN (46) ranks third, being much less accurate than the other two methods. This is primarily the consequence of using an approximation for the arcsin function. However, it still manages to match the first 2 digits of the web integrator’s result, which is an incredible result for an approximation formula.

Discussion

Not unexpectedly, the two numerical integration methods, when applied to the *true* distortion expression, i.e., with the arcsin function included, provide results which are *much* more accurate than the results obtained by integrating Löfgren's approximation to the WTE (i.e., without the arcsin function). Even so, the accuracy of Löfgren's EQN (46) is most impressive, being that it manages to achieve two digits of accuracy at a time when log tables and slide rules were the calculating tools of the day! How good is that!

THE "DIFFICULT" ARCSIN FUNCTION IN LÖFGREN'S EQN (22)

Tracking distortion is given by Löfgren's EQN (22):

$$\varepsilon = \frac{V}{\Omega} \cdot \frac{\delta}{R}$$

$$= \frac{V}{\Omega} \cdot \text{WTE}$$

$$= \frac{V}{\Omega} \cdot \left[\frac{\arcsin\left(\frac{R^2 + L^2 - M^2}{2LR}\right) - \beta}{R} \right] \qquad (1)$$

To calculate the RMS distortion, we need to integrate the square of EQN (1) with respect to the radius R. However, the arcsin function causes a problem:

$$\arcsin\left(\frac{R^2 + L^2 - M^2}{2LR}\right)$$

$$= \arcsin\left(\frac{R}{2L} + \left(\frac{L^2 - M^2}{2L}\right) \cdot \frac{1}{R}\right)$$

$$= \arcsin\left(\frac{R}{a} + \frac{b}{R}\right) \qquad (2)$$

where $a = 2L$, $b = \frac{L^2 - M^2}{2L}$

and a, b are constants.

EQN (2) does not have a known closed form integral with respect to radius R.

ALIGNMENT FOR *TRACKING ERROR* MINIMISATION (BEFORE 1938)

1. Background

The cutting of a record master on a cutting lathe takes place with the vertical plane through the effective longitudinal axis of the vibrating stylus being positioned at right angles to a line drawn between the stylus tip and the record centre. This arrangement is applied at all positions of the cutting stylus across the record surface during the cutting of the disc.

When the finished record is later played on a turntable, we need the vertical plane through the stylus cantilever to similarly be at right angles to the line drawn between the stylus tip and the record centre at all playing positions across the record surface. The positioning of the playback stylus in this manner ensures the motion imparted to the stylus is a faithful replica of the motion of the recording lathe cutting stylus, maximising the fidelity of the reproduced signal and minimising the wear on the record groove walls.

2. The Pivoted Tonearm

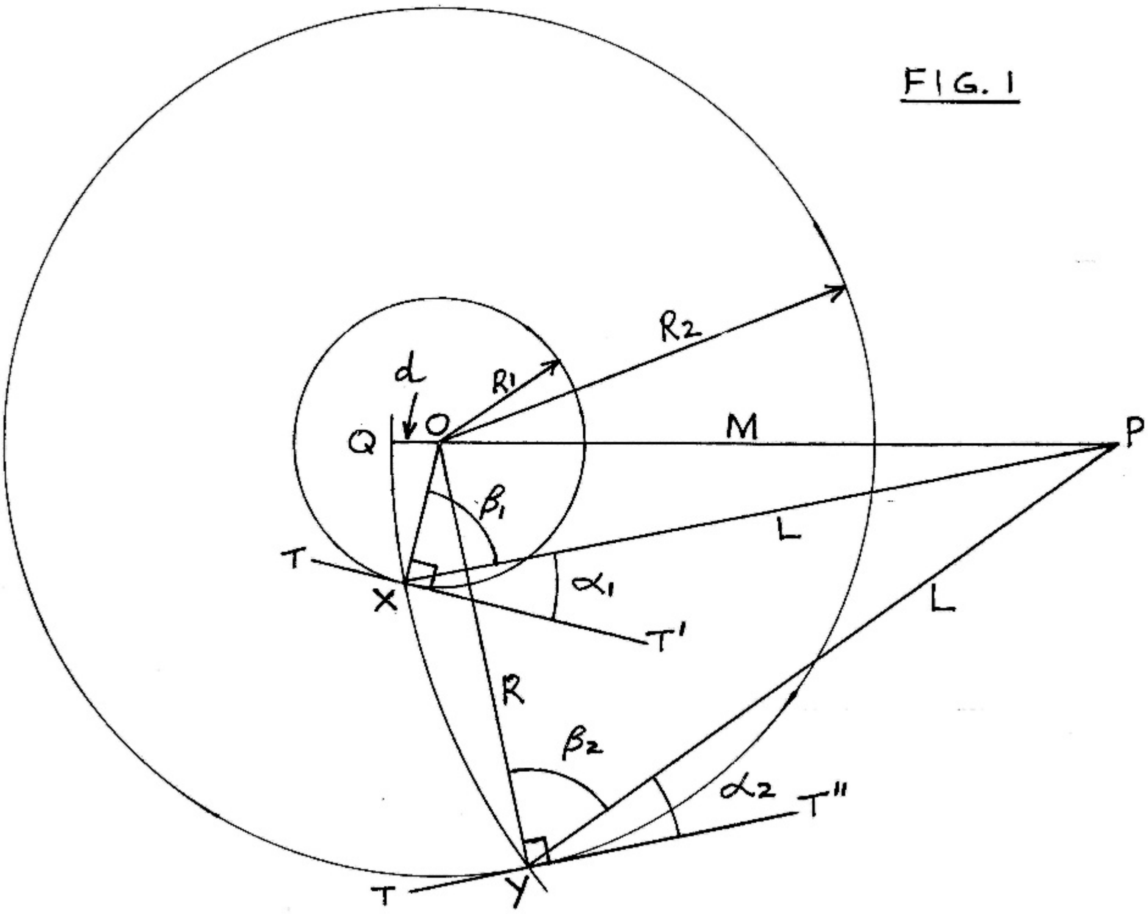
Consider the pivoted tonearm. With this arrangement, the motion of the playback stylus across the playing surface cannot replicate the motion of the original cutting stylus because a line drawn through the stylus cantilever cannot always be at right angles to a line drawn between the stylus tip and the record centre at all playing positions. The consequence is that the sound fidelity is impacted by the resulting distortion, and the groove walls suffer additional wear from the continual rotating/grinding/reshaping action of the stylus. The aim of lateral tracking error minimisation is to reduce these consequences to the lowest possible levels.

Groove damage is also possible by another means with non-optimum alignment. Variation in tracking *angle* normally occurs smoothly across a disc's playing surface. It might easily happen however that the sudden change in tracking angle between the inner grooves of the record which has finished playing to the outer grooves of the next record, and should the needle tip be worn and the needle is not replaced, it could result in the needle presenting a sharp, cutting chisel edge to the groove walls in the first grooves of the next record, creating additional groove damage. Optimum design eliminates this problem by ensuring the tracking angles at the beginning and end radii are identical, eliminating this groove damage problem.

3. Bela Harsanyi's 1908 Patent

We will now consider in detail the analysis and results by Bela Harsanyi in his 1908 French patent. Consider a pivoted tonearm mounted on a turntable. Also consider for the moment that *no offset angle has been applied,* i.e. the stylus cantilever is in line with a line drawn between the stylus tip and the tonearm pivot. This creates tracking error because the stylus cantilever is not at right angles to the line drawn between the stylus tip and the centre of the record.

Referring now to Fig. 1 below, P is the location of the tonearm pivot, O is the turntable centre, PQ = L = the effective length of the tonearm, comprised of the mounting distance PO (M) and the overhang OQ (d). X is the stylus tip location when at the inner groove R_1, and Y is the stylus tip location when at the outer groove R_2. OX is a radius at R_1 and TT′ is a line passing through X at right angles to OX. OY is a radius at R_2 and TT′′ is a line passing through Y at right angles to OY. The circle arc YXQ is of radius L, centred at P, and is the path traced by the stylus tip between the outermost groove at Y and the innermost position Q, passing through X.



When the stylus tip is at X, the tracking angle α_1 is the angle PXT′, which is the angle between the stylus cantilever (on line PX) and the line TT′. When the stylus tip is at Y, the tracking angle α_2 is the angle PYT′′, which is the angle between the stylus cantilever (on line PY) and the line TT′′. As no offset angle has been applied, the angle α_1 is the tracking error at X, and the angle α_2 is the tracking error at Y.

In the triangles OXP and OYP, for a specified L, R_1 and R_2, there will be a particular position of the tonearm pivot P which results in a particular overhang d (OQ) and mounting distance M (PO), which makes angles OXP (β_1) and OYP (β_2) equal. At the same time, angles PXT′ (α_1) and PYT′′ (α_2) will also be equal, α and β being complimentary. This situation is depicted in Fig. 1.

Tracking Angle α
At a radius R, we may determine the tracking angle α as follows:
From a triangle, say POY in Fig. 1, then from the Cosine Rule:
$M^2 = R^2 + L^2 - 2LR \cos \beta$ (where $\beta = \beta_2$).
But as $\cos \beta = \sin \alpha$ (β and α are complementary), then
$M^2 = R^2 + L^2 - 2LR \sin \alpha$, or
$\sin \alpha = (R^2 + L^2 - M^2) / (2LR)$ (1)
so $\alpha = \text{ASIN}((R^2 + L^2 - M^2) / (2LR))$ (2)

EQN (2) for the tracking angle α is used for generating the tracking angle plots in Fig. 2 and Fig. 3 following.

4. Distortion and Wear

For the pivoted tonearm, for any position of the stylus across the playing surface between R_2 and R_1, if the tracking error is not zero, distortion of the recorded sound occurs. At the same time, wear of the record groove walls occurs due to the rotating motion of the stylus in the groove as the stylus traverses the record surface, caused by the (wear of the) leading edge of the stylus twisting and chiselling its way into the groove walls as the record is played.

Although *any* tracking error results in distortion and any (fixed) tracking error results in the wear of one groove wall more than the other, it is not the *magnitude* of the tracking error which causes the real wear of the grooves, but rather it's the *change* in the tracking error which occurs between the outer and inner radii during play. This *change* in tracking error causes the stylus to rotate about its point of contact with the groove during play and produce wear of the groove walls. (The stylus on a radial tracking arm does not rotate in this manner because the tracking error remains fixed at zero degrees at all groove positions across the playing surface. Thus no rotation of the stylus occurs, so no groove wall damage occurs by this means.)

To this point, offset angle has not been applied.

Historically, tonearms mounted on the earliest (acoustic) gramophones were typically mounted with zero overhang, meaning the stylus tip would pass through the *centre* of the record when the arm was swung across the record.

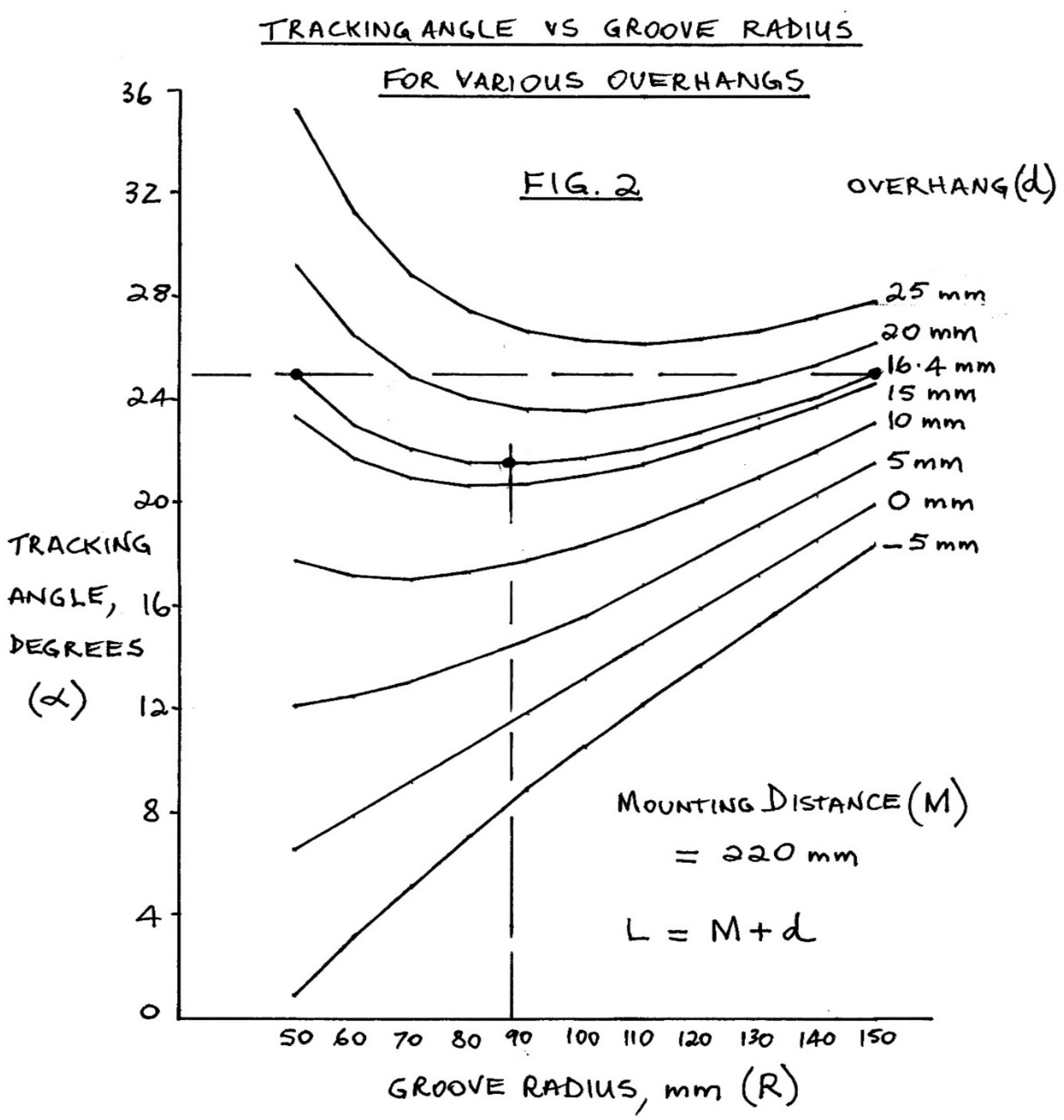
At some unknown (to this writer) point in history, the concepts of offset angle and overhang were first originated as solutions to *minimising* distortion and record wear. The identity of the originator is not known to the writer. At those earlier times, it is possible that multiple people, independently of each other, devised offset angle and overhang concepts, perhaps over a period of years, and perhaps without knowledge of each other or each other's work. Perhaps we may never know who first devised these concepts in chronological terms.

5. Tracking Angle Plots

We'll firstly consider how the application of overhang affects the tracking angle across the playing surface of the record. Refer Fig. 2.

Using the formula for tracking angle at EQN 1, we will calculate and plot the tracking angle α between the groove radii R_1 = 50 mm and R_2 = 150 mm for a mounting distance M = 220 mm, over a range of underhang (negative overhang) and overhang values to show how these values affect the tracking angle plots and the changes in the tracking angle. The initial aim is to select the overhang which *minimises the change* in tracking angle across the record playing surface, regardless of its magnitude.

Looking at Fig. 2, it shows plots of the tracking angles for eight overhang values. The plot for the -5 mm overhang option is for an arm length of 215 mm, which is 5 mm less than the mounting distance (220 mm), so the stylus underhangs the record centre by 5 mm. The plot for 0 mm is for when the arm length equals the mounting distance (220 mm), where the stylus is able to swing on a curve which passes through the centre of the record. The remaining plots are for overhang values for the numbers shown.



6. Step 1: Minimise the *change* in tracking angle to minimise record wear

In Fig. 2, it may be observed the plot with an underhang of 5 mm (the -5 mm plot) shows a *change* in tracking angle of around 17 degrees across the radii R_1 to R_2. For the other plots, the *change* in tracking angle continues to reduce to a minimum of around 3.5 degrees when the overhang is at 16.4 mm, where the tracking angles at R_1 to R_2 are equal, then the *change* begins to increase to around 9 degrees at an overhang of 25 mm.

From Fig. 2, it may be observed that the plot for an overhang of 16.4 mm produces a maximum tracking angle of 25 degrees at both the 50 mm and the 150 mm groove radii, and a minimum tracking angle of approx. 21.5 degrees at around 87 mm groove radius. For this overhang setting, the *change* in tracking angle over the 50 mm to 150 mm radii is 25 - 21.5 = 3.5 degrees as noted above. All other plots have greater changes in the tracking angle.

In the goal to minimise the *change* in tracking angle across the record playing surface as the first optimisation step, a useful mathematical approach is to *equalise* the tracking angles at the two radii R_1 and R_2. When this is achieved, it will in turn minimise the *change* in tracking angle between the largest and smallest tracking angles.

7. Solving for Optimum Overhang:

The overhang required to equalise the tracking angles at R_1 and R_2 may be determined as follows. As we require these two tracking angles to be equal, we use EQN (1) for each of these radii in turn, then equate them, then solve the expressions for the overhang. Refer to para. 10 following for the derivation of the overhang required to set the tracking angle at R_1 and R_2 equal.

In summary, for the example data and plots shown, the overhang of 16.4 mm with a mounting distance of 220 mm, i.e. with an effective arm length of 236.4 mm, produces the ideal alignment to minimise the *change* in tracking angle across the groove radii range. This in turn *minimises the wear* on the record groove walls by minimising the rotating motion of the stylus as it crosses the playing surface, addressing the first of the two problems which occur when using the pivoted tonearm – record wear.

8. Step 2: Minimise the tracking error to minimise distortion

When the tracking angle is zero, i.e. when a line through the stylus cantilever is at right angles to a line from the stylus tip to the record centre, tracking distortion is zero because the stylus faithfully follows the original motion of the cutting stylus. However, when a pivoted arm is used, zero tracking angle cannot be maintained at all points between the outer and inner radii.

It will now be shown how the application of an offset angle minimises the distortion by minimising the *magnitude* of the tracking error. Remember, whenever we have non-zero tracking error, we have distortion of the reproduced sound.

Referring to Fig. 2 and the plot for the 16.4 mm overhang setting. As previously noted, the tracking angle swings between the maximum limit of 25 degrees at the inner and outer groove radii and the minimum limit of 21.5 degrees at 87 mm radius, a change in tracking angle of 3.5 degrees.

The strategy is to apply an offset angle which offsets the tracking angle plot. Consider the 16.4 mm overhang plot in Fig. 2, where the tracking angle is 25 degrees at the inner and outer radii. Referring to Fig. 1, if we were to rotate/skew the cartridge body clockwise by that angle, 25 degrees, then at the inner and outer groove radii, points X and Y respectively, the stylus cantilever would be in line with the lines TT' and TT'' respectively so tracking error would be zero at those points and tracking distortion would be zero. Refer to para. 11 following for the derivation of the offset angle required to set the tracking error at R1 and R2 to zero.

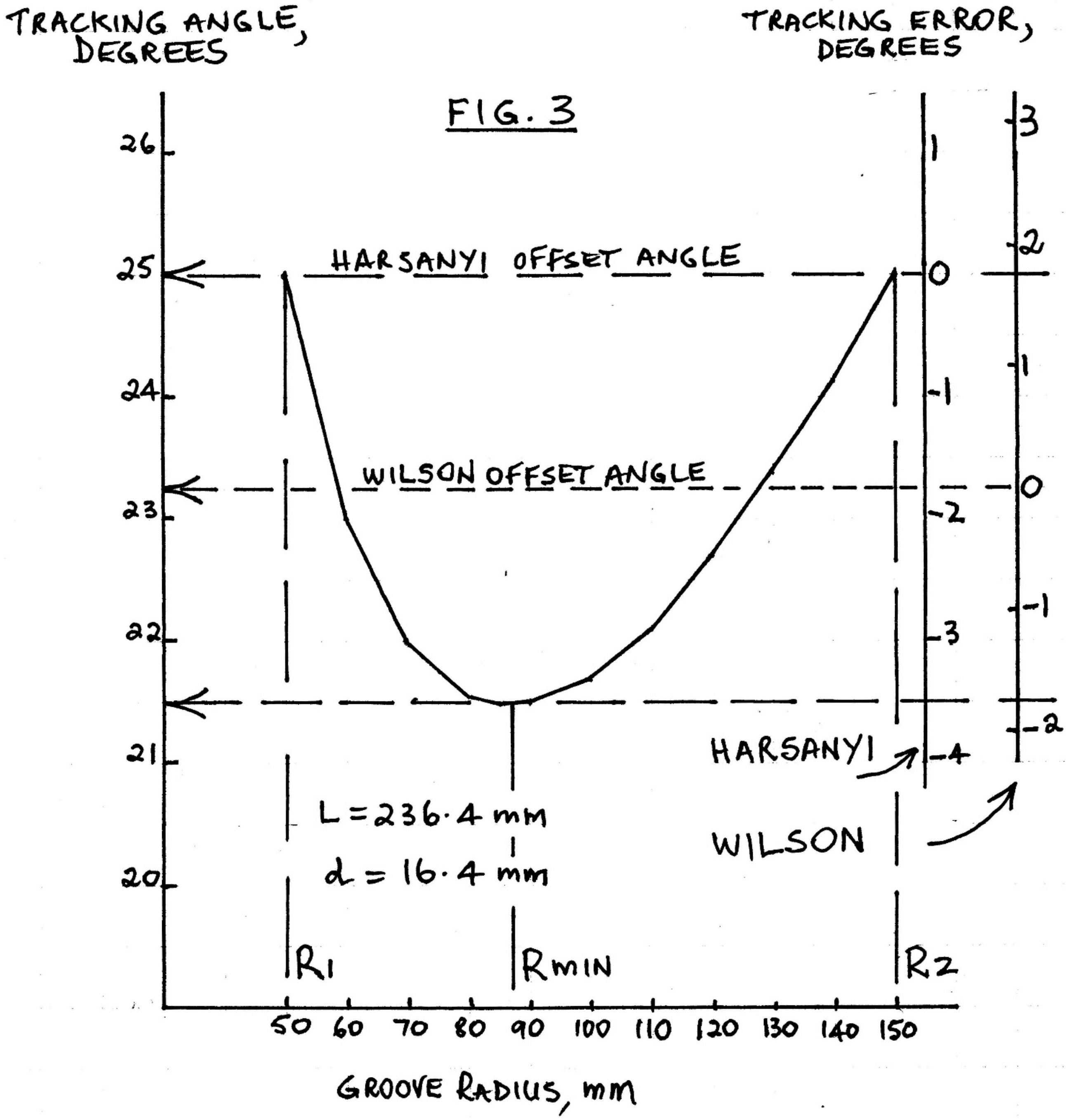
Having applied the 25 degrees offset angle, we note the tracking error starts at zero at the outer radius, then it slowly increases to the maximum *magnitude* of 3.5 degrees at the curve minimum at 87 mm radius, then slowly reduces to zero at the inner radius.

This is the strategy devised by Bela Harsanyi in his 1908 French patent.

9. Tracking Angle and Tracking Error Plot

Fig. 3 provides an expanded plot of the 16.4 mm overhang arrangement of Fig. 2. It shows a range of tracking angles from 20 degrees to 26 degrees across the same groove radii (50 mm to 150 mm). It may be noted that a horizontal line has been drawn at the 25-degree tracking angle, labelled *Harsanyi Offset Angle*, which intercepts the plot at the 50 mm and 150 mm radii points. It may also be observed that the minimum of the plot occurs at around 87 mm, where the tracking angle is around 21.5 degrees. (All as previously described.)

Fig. 3 also includes a scale on the right side, zero-centred at the 25-degree tracking angle line, labelled *Harsanyi*. In effect, when an offset angle of 25 degrees is applied, the tracking error is zero at the inner and outer radii as shown on the right-side scale. The tracking error resulting at any other radius across the record surface may also be determined, reaching a maximum magnitude of 3.5 degrees at 87 mm radius.



10. SOLVE FOR OPTIMUM OVERHANG

Applying EQN (1) and solving for overhang d:

$\sin\alpha_1 = (R_1^2 + L^2 - M^2)/(2LR_1)$ and

$\sin\alpha_2 = (R_2^2 + L^2 - M^2)/(2LR_2)$ so we set

$\sin\alpha_1 = \sin\alpha_2$, or

$$\text{ie}\quad \frac{R_1^2 + L^2 - M^2}{2LR_1} = \frac{R_2^2 + L^2 - M^2}{2LR_2}$$

$$\text{ie}\quad \frac{R_1^2}{2LR_1} + \frac{L^2 - M^2}{2LR_1} = \frac{R_2^2}{2LR_2} + \frac{L^2 - M^2}{2LR_2}$$

$$\text{ie}\quad \frac{L^2 - M^2}{2LR_1} - \frac{L^2 - M^2}{2LR_2} = \frac{R_2^2}{2LR_2} - \frac{R_1^2}{2LR_1}$$

$$\text{ie}\quad \frac{L^2 - M^2}{2L}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{1}{2L}\left[R_2 - R_1\right]$$

$$\text{ie}\quad (L^2 - M^2)\left(\frac{R_2 - R_1}{R_2 R_1}\right) = R_2 - R_1$$

$$\text{ie}\quad (L^2 - m^2)\frac{1}{R_1 R_2} = 1$$

$$\text{ie}\quad L^2 - M^2 = R_1 R_2 \qquad (3)$$

But $M = L - d$

so $L^2 - (L - d)^2 = R_1 R_2$

ie $L^2 - (L^2 - 2Ld + d^2) = R_1R_2$

ie $L^2 - L^2 + 2Ld - d^2 = R_1R_2$

ie $2Ld - d^2 = R_1R_2$

and $d^2 - 2Ld + R_1R_2 = 0$

So $$d = \frac{2L \pm (4L^2 - 4R_1R_2)^{\frac{1}{2}}}{2}$$
$$= \frac{2L \pm 2(L^2 - R_1R_2)^{\frac{1}{2}}}{2}$$

ie $d = L \pm (L^2 - R_1R_2)^{\frac{1}{2}}$

TAKING THE SMALLER SOLUTION, OVERHANG d is:

$$d = L - (L^2 - R_1R_2)^{\frac{1}{2}} \qquad (4)$$

USING THIS OVERHANG WILL RESULT IN THE TRACKING ANGLES AT R1 AND R2 BEING EQUAL AS WE REQUIRE, MINIMISING THE CHANGE IN TRACKING ANGLE ACROSS THE PLAYING SURFACE WHICH MINIMISES RECORD WEAR.

EQN (4) WAS PROVIDED BY HARSANYI.

11. SOLVE FOR OPTIMUM OFFSET ANGLE

APPLYING EQN (1) AND SOLVING FOR MAXIMUM TRACKING ANGLE (AT R_1, R_2):

LET $R = R_1$, then

$$\sin \alpha_1 = \frac{R_1^2 + L^2 - M^2}{2LR_1}$$

FROM EQN (3): $L^2 - M^2 = R_1 R_2$

$$\text{SO } \sin \alpha_1 = \frac{R_1^2 + R_1 R_2}{2LR_1}$$

$$= \frac{R_1 + R_2}{2L}$$

THEREFORE THE MAXIMUM TRACKING ANGLE AT R_1, R_2

$$= \text{ASIN}\left(\frac{R_1 + R_2}{2L}\right) \qquad (5)$$

SETTING THE OFFSET ANGLE TO THIS VALUE WILL RESULT IN ZERO TRACKING ERROR AT R_1, R_2.

EQN (5) WAS PROVIDED BY HARSANYI.

12. SOLVE FOR THE MINIMUM OF THE TRACKING ANGLE CURVE

THE MINIMUM OF THE TRACKING ANGLE CURVE OCCURS AT THE RADIUS Rmin, WHERE THE DERIVATIVE OF THE TRACKING ANGLE EQUATION IS ZERO.

FROM EQN (1):

$$\sin \alpha = \frac{R^2 + L^2 - m^2}{2LR}$$

$$= \frac{R^2}{2LR} + \frac{L^2 - m^2}{2LR}$$

$$= \frac{1}{2L} \cdot R + \frac{L^2 - m^2}{2L} \cdot \frac{1}{R}$$

THEN $\frac{d}{dR} \sin \alpha = \frac{1}{2L} - \frac{L^2 - m^2}{2L} \cdot \frac{1}{R^2}$

$$= \frac{2LR^2 - 2L(L^2 - m^2)}{(2LR)^2}$$

$$= \frac{2LR^2 - 2L^3 + 2Lm^2}{(2LR)^2}$$

$$= \frac{2L(R^2 - L^2 + m^2)}{(2LR)^2} \qquad (6)$$

EQUATE EQN (6) TO ZERO AND SOLVE FOR R = Rmin, THE RADIUS OF THE MINIMUM TRACKING ANGLE POINT ON THE TRACKING ANGLE CURVE.

FOR EQN (6) TO BE ZERO, AND KNOWING L AND R ARE NOT ZERO, THEN

$$R^2 - L^2 + m^2 = 0$$

SO $$R^2 = L^2 - m^2$$

AND $R = R_{MIN} = \left(L^2 - M^2\right)^{\frac{1}{2}}$ (7)

EQN (7) IS THE RADIUS AT WHICH THE TRACKING ANGLE REACHES ITS MINIMUM, AND IS EXACT, I.E. WITHOUT APPROXIMATION.

FROM EQN (2), TRACKING ANGLE AT RMIN

$$\alpha_{MIN} = \text{ASIN}\left(\frac{R_{MIN}^2 + L^2 - M^2}{2L R_{MIN}}\right)$$

SO FROM EQN (7):

$$\alpha_{MIN} = \text{ASIN}\left(\frac{\left(L^2 - M^2\right) + L^2 - M^2}{2L\left(L^2 - M^2\right)^{\frac{1}{2}}}\right)$$

$$= \text{ASIN}\left(\frac{2L^2 - 2M^2}{2L\left(L^2 - M^2\right)^{\frac{1}{2}}}\right)$$

$$= \text{ASIN}\left(\frac{2\left(L^2 - M^2\right)}{2L\left(L^2 - M^2\right)^{\frac{1}{2}}}\right)$$

ie $$\alpha_{MIN} = \text{ASIN}\left(\frac{\left(L^2 - M^2\right)^{\frac{1}{2}}}{L}\right) \quad (8)$$

= TRACKING ANGLE AT RMIN

FROM EQN (3), AS $L^2 - M^2 = R_1 R_2$

then $$\alpha_{MIN} = \text{ASIN}\left(\frac{\left(R_1 R_2\right)^{\frac{1}{2}}}{L}\right) \quad (9)$$

= TRACKING ANGLE AT RMIN

13. HARSANYI MAXIMUM TRACKING ERROR

WITH THE OVERHANG SET PER EQN (4) AND THE OFFSET ANGLE SET PER EQN (5), THE TRACKING ERROR IS ZERO AT R_1, R_2 AND GREATER IN MAGNITUDE AT ALL INTERMEDIATE RADII.

THE TRACKING ERROR REACHES ITS GREATEST MAGNITUDE AT R_{MIN}, AND IS GIVEN BY:

$$\left| \text{TRACKING ANGLE AT } R_1 - \text{TRACKING ANGLE AT } R_{MIN} \right|$$

$$= \left| \text{EQN (5)} - \text{EQN (9)} \right|$$

$$= \left| \text{ASIN}\left(\frac{R_1 + R_2}{2L}\right) - \text{ASIN}\left(\frac{(R_1 R_2)^{\frac{1}{2}}}{L}\right) \right| \qquad (10)$$

14. Summary of Bela Harsanyi's Results

In summary, Harsanyi chose an overhang so that the tracking angles at the outer and inner radii are made equal, minimising the *change* in tracking angle across the playing surface, and in so doing, minimised the rotational wear on the record groove walls by the stylus.

Harsanyi then chose an offset angle so that the tracking errors at the outer and inner radii are zero, which then provides for a maximum *magnitude* of tracking error of 3.5 degrees at around the centre of the playing surface.

In his patent, Harsanyi observes that an even better outcome could be achieved if the offset angle was set to a determinable angle between the extremes of the tracking angle curve, resulting in the tracking error being even lower. Although Harsanyi was the first to make this observation, no formula or discussion was presented for calculating the improved offset angle in any precise terms.

Harsanyi's work is of historical significance, being that he was the first person to apply an analytical analysis to tonearm alignment optimisation to minimise tracking distortion and record wear.

15. Louis Lumière's 1911 Patent

Louis Lumière presented a geometrical solution to minimise the distortion and record wear resulting from tracking error. His criteria and geometrical description of overhang and offset angle *principles* are identical to Harsanyi's, but formulas were not presented.

In Lumière's patent, he observes that a better outcome could be achieved by setting the offset angle to an angle between the tracking angle extremes, resulting in the tracking error being even lower, just as Harsanyi had stated. This would cause the two points *m* and *n* in his Fig. 4 to be drawn closer together, such as to his points *i* and *j*. Also, just like Harsanyi, he failed to specify how the improved offset angle should be calculated in any precise terms.

16. Percy Wilson's 1924 Article

Percy Wilson applied the *identical* overhang criteria and formula as used by Harsanyi to minimise record wear by equalising the tracking angles at the inner and outer radii as shown in Fig. 3.

However, Wilson applied a different offset angle criteria and formula. Wilson observed that if the offset angle is centred between the tracking angle extremes at R_1, R_2 and at R_{MIN}, the maximum tracking error is given by:

$$\left| \frac{\text{TRACKING ANGLE AT } R_1 - \text{TRACKING ANGLE AT } R_{MIN}}{2} \right|$$

$$= \left| \frac{\text{EQN }(5) - \text{EQN }(9)}{2} \right|$$

$$= \frac{1}{2} \left| \text{ASIN}\left(\frac{R_1 + R_2}{2L}\right) - \text{ASIN}\left[\frac{(R_1 R_2)^{\frac{1}{2}}}{L}\right] \right|$$

THIS IS HALF THE MAXIMUM TRACKING ERROR PROVIDED BY HARSANYI.

WILSON'S OFFSET ANGLE

AS NOTED ABOVE, WILSON SET THE OFFSET ANGLE 'MIDWAY' BETWEEN THE EXTREMES OF THE TRACKING ANGLE CURVE. THE OFFSET ANGLE IS SET TO:

$$= \left| \frac{\text{TRACKING ANGLE AT } R_1 + \text{TRACKING ANGLE AT } R_{MIN}}{2} \right|$$

$$= \left| \frac{\text{EQN }(5) + \text{EQN }(9)}{2} \right|$$

$$= \frac{1}{2} \left| \text{ASIN}\left(\frac{R_1 + R_2}{2L}\right) + \text{ASIN}\left[\frac{(R_1 R_2)^{\frac{1}{2}}}{L}\right] \right|$$

In Fig. 3, where the plot shows a maximum change in tracking angle magnitude of 3.5 degrees as already discussed, Wilson set the offset angle to 23.25 degrees, as shown in Fig. 3, **exactly in the centre of the tracking angle extremes,** as provided by his offset angle formula above. Wilson's offset angle formula is perfectly correct. Wilson's tracking error scale on the right side of Fig. 3 is zero-centred at 23.25 degrees and shows that Wilson's maximum tracking error is ± 1.75 degrees, *exactly half* the maximum tracking error of 3.5 degrees provided by Harsanyi. It is not known if Wilson knew of Harsanyi's 1908 patent.

17. Lofgren's 1929 Article

This article was translated into English by Klaus Rampelmann in 2021. Thank you once again Klaus for your contribution to the history of tonearm alignment and its access to the readers who are language-challenged like this writer!

18. Löfgren's Overhang

The criteria and formula for the overhang are *identical* to Harsanyi's and Wilson's.

19. Löfgren's Offset Angle

The criteria for the offset angle is *identical* to Wilson's, but Löfgren developed an alternative formula for the offset angle which is an approximation. The offset angle produced by his formula is *very close to*, but is not exactly the same as, Wilsons.

Löfgren set the offset angle to:

$$\mathrm{ASIN}\left(\frac{R_1 + R_2 + 2(R_1 R_2)^{\frac{1}{2}}}{4L}\right)$$

20. Summary Of Tracking Error Minimisation (Before 1938)

The works of Harsanyi (1908), Lumière (1911), Wilson (1924) and Löfgren (1929) have a common purpose: to minimise lateral tracking error to minimise distortion of the reproduced sound and to minimise wear of the record.

All four authors set about this by defining an overhang to minimise the *change* in tracking angle and the resulting record wear, then defining an offset angle to minimise the *magnitude* of the tracking error and the resulting distortion of the reproduced sound.

21. Optimum Overhang

Harsanyi, Lumière, Wilson and Löfgren used overhang figures which set the tracking angles at R_1 and R_2 equal. We know the overhang formulas presented by Harsanyi, Wilson and Löfgren are identical, while Lumière did not present an overhang formula, but rather stated that the tracking angles at R_1 and R_2, his points m, n in Figs. 3 and 4 of his patent, are made equal, implying the tracking angles at R_1 and R_2 are equal. No differences there.

22. Optimum Offset angle

Harsanyi and Lumière set the offset angle equal to the tracking angle at R_1 and R_2, which are at the maximums of the tracking angle curve.

Wilson and Löfgren set the offset angle to the *centre* of the tracking angle extremes, providing the optimum offset angle, where the tracking error is *half* the error provided by Harsanyi's and Lumière's results.

23. True Zero Tracking Error (Null) Radii

The zero tracking error (null) radii are the radii at which the tracking angle curve is intersected by the offset angle line per Fig. 3, and are therefore the radii where the tracking error is zero. None of the pre-1941 articles discussed the calculation of the null radii. Thus,

tracking error = tracking angle – offset angle (β)

$$= \mathrm{ASIN}\left(\frac{R^2 + L^2 - M^2}{2LR}\right) - \beta$$

Mathematically, the null radii are the roots of the tracking error formula above when it is equated to zero and solved for the radius R to give the null radii R_{01} and R_{02}. (Note that the offset angle and overhang do not have to be optimum.) The null radii are then given by:

$$R_{01}, R_{02} = L\sin\beta \pm \left[L^2\sin^2\beta - (L^2 - M^2)\right]^{\frac{1}{2}}$$

24. Approximate Zero Tracking Error (Null) Radii

A recent article by Vladan Jovanovic [17] presents a new, approximate formula for the null radii in terms of the groove radii R_1 and R_2 when Wilson's optimum offset angle and overhang formulas are employed to minimise tracking error. The null radii, R_{01} and R_{02}, given by the following formula, are *exceptionally* close to the true null radii calculated by the formula above. This is a new result in tonearm alignment history.

$$R_{01,02} = \frac{\left(\sqrt{R_1} + \sqrt{R_2}\right)^2}{4} \pm \sqrt{\frac{\left(\sqrt{R_1} + \sqrt{R_2}\right)^4}{16} - R_1 R_2}$$

25. Further Developments

Over the history of tonearm alignment optimisation up to 1938, numerous articles were published on the subject for purposes of maximising the fidelity of the reproduced sounds and minimising the wear of the record grooves, although *none of the articles made any changes to the findings and results provided by Wilson in 1924.*

26. The Final Development!

Historically, tonearm alignment optimisation was forever changed in 1938 (although it took until 1983 for it to be recognised globally) following the publication of an article by Löfgren which showed that *weighted tracking error* is the parameter to be minimised to minimise distortion, not *tracking error*. No further optimised alignment *developments* have been made since that historic article. Löfgren's 1938 article stands as having solved the alignment problem once and for all. His work was the trigger behind this book.

FACTS AND QUESTIONS ABOUT BAERWALD'S ARTICLE

Background

Firstly, consider Erik Olof Löfgren: 23 February 1897 - 7 August 1986. Born in Södermanland County in Sweden. He was an engineer and professor of radio technology at the Royal Institute of Technology in Stockholm, Sweden (KTH). His 1938 tonearm article was written in German and published in a new (less than three years old), little-known Nazi-Germany journal. Löfgren's article includes his EQN (22) and its derivation, which shows the distortion resulting from tracking error. It unifies the associated fundamental parameters, and to the best of the writer's knowledge, he was the first to show this relationship. This equation is one of Löfgren's finest achievements. The detailed derivation of his EQN (22) is shown elsewhere in this analysis. Löfgren also shows the derivation of his EQN (33), which is an excellent approximation of the WTE. This equation was introduced to simplify the mathematical analysis by removing the "difficult" inverse sine function. Löfgren solves the optimum offset angle and optimum overhang equations *directly* from his EQN (33). We refer to these two equations as the 'Löfgren A' alignment equations. The detailed derivation of Löfgren's EQN (33) and the optimum alignment equations are included elsewhere in this analysis. In fact, eighteen of Löfgren's equations are derived in full detail from first principles in Sections S10 and S11, which the writer believes is the first time the derivation of these equations has been presented. (The writer originally derived these in 1983). Löfgren's findings and results are based on his EQNs (22) and (33), and as a pair of equations, are at the core of his work.

Next, consider Hans Georg Baerwald: 5 April 1904 - 27 February 1987. Born in Breslau, (then in) Germany. Received Doctorate of Engineering degree from the Technical University of Breslau in 1930. Emigrated to USA in 1939. German was his first language. As Löfgren's article was written in German, Baerwald would have understood it perfectly from a linguistic sense and from a mathematical sense. There is no doubt in this writer's mind that Baerwald knew and understood the mathematics of Löfgren's article in the finest of academic detail. Baerwald not only cites Löfgren's work as his third reference, but he specifically refers to it on three separate occasions. As a clearly confirming example of his familiarity with and understanding of Löfgren's article, consider the detail in the footnote on page 600 of Baerwald's article. Time, distance and language played no part whatsoever in how Baerwald viewed and understood Löfgren's article.

By the time Baerwald's article was published in 1941, he was an eminent scholar and published engineer. He would have understood and respected Löfgren's article for its historical significance and for the insight shown by Löfgren. As a researcher, Baerwald would be keen to ensure the reader understood the key steps in his analysis which underpinned his findings, and as a senior scholar, he would know the accepted conventions of acknowledgement and attribution.

Now consider Baerwald's article. Baerwald introduces the second-order Chebyshev approximation method on page 608 as the method used to develop his alignment solution and equations. On page 609, he states after the sub-heading at sub-para (c): "The second-order Chebyshev approximation gives the following optimal values:", then presents his alignment equations at EQN (16). Baerwald provides no evidence of the method of derivation of these equations, nor of the manner in which the Chebyshev method was applied. (Note that the use of the Chebyshev method is *not* essential for the derivation of the 'Löfgren A' equations, all of which can be developed from first principles, as shown by Löfgren, and as shown by this writer elsewhere in this analysis.)

The Alignment Equations

In the following table, the first column lists seven of the defining equations associated with the 'Löfgren A' alignment solution. The second and third columns show the equation numbers of each of those equations as they appear in their respective authors' articles.

Table Of 'Löfgren A' Alignment Equations

'Löfgren A' Equations	Löfgren's EQNs	Baerwald's EQNs	Where Proven here?
Tracking Distortion	22	10	Page S12-37
Weighted tracking error	33	14	Page S12-31
Optimum offset angle	34	16a	Page S8-2
Optimum overhang	39	16b	Page S8-3, S8-4
Radius R_a^2	40	part of 16b	Page S12-34
Radius R_w (general)	36	18	Page S12-35
Radius R_w (optimum)	37	part of 16e	Page S12-38

The Facts

In 1983, through a letter published in *Audio* [7], this writer advised the fact that Baerwald's (and others') optimum offset angle and overhang equations are *identical* to the equations presented by Löfgren (apart from notation and arrangement).

In fact, *all seven* of Baerwald's equations listed in column 3 of the table above are *identical* to Löfgren's equations listed in column 2 (apart from notation and arrangement).

The fourth column indicates where this is proven in this document.

None of Baerwald's equations in the table above are original to Baerwald, a fact that Baerwald would have absolutely known because of his detailed knowledge and understanding of Löfgren's article. However, on reading Baerwald's article, one could be forgiven for concluding that Baerwald originated these equations because of a complete absence of attribution to Löfgren for the equations.

Like Baerwald, the later authors - Bauer, Seagrave, Stevenson and Kessler and Pisha, also provided optimum alignment equations which are identical or approximately identical to Löfgren's, as the writer reported in 1983, and as shown on page S8-1 of this analysis. However, unlike Baerwald, these authors discussed their respective optimum solutions and explained the basis of the derivation of their formulas. Baerwald provided no such discussion of his published equations nor provided any details about their derivation. Of course, there was no requirement for him to do so, but it is a most unusual stance for a senior scholar like Baerwald.

The Unanswered Questions

As the writer's knowledge is limited to the information contained in Baerwald's article, some concerning questions arise, including:

1. Why did Baerwald choose to deny attribution to Löfgren for Baerwald's equations listed in column 3 of the table above?

2. Why did Baerwald choose not to show the background analysis to and the derivation of his equations listed in column 3?

3. Why did Baerwald introduce discussion of the Chebyshev method, knowing its application is *not* essential for the derivation of his equations in the table, and while knowing his equations are identical to Löfgren's apart from notation and arrangement?

4. If the Society of Motion Picture Engineers, which published Baerwald's article, had peer review protocols in place, why weren't these issues identified, after which Baerwald would have been able to resolve them in accordance with the SMPE's publication and standards requirements prior to the article's publication, after which these issues would never have arisen?

Further Insights

After its publishing in 1941, Baerwald's article was received in exalted terms by recorded music enthusiasts, because Baerwald's rigorous and extensive analysis had, at long last, identified the basic mathematical relationships linking the parameters involved in the generation of tracking distortion which researchers had been seeking for many years. Baerwald had identified the origin, nature and impact of tracking distortion, he showed that tracking distortion is proportional to the WTE, and he presented alignment equations for calculating optimum offset angle and overhang values which would minimise tracking distortion. By applying these results, tracking distortion was able to be minimised, greatly enhancing listening pleasure. The fact that Löfgren had already developed and published *identical* results three years earlier was clearly known to Baerwald. Perhaps no one else on the planet, apart from Baerwald and Löfgren, was any the wiser.

Thus, from the time of publishing of Baerwald's article in 1941, up to the time of the publishing of this writer's letter to *Audio* in May 1983, the audio world recognised and credited Baerwald with being the first person to develop the optimum alignment equations. Perhaps this reality could have been assisted by Baerwald's article being published in English in a world-recognised and highly respected journal, while Löfgren's 1938 article was published in German in a new, little-known Nazi-Germany acoustic journal during a period of increasing instability in Europe. This may well have contributed to Löfgren's article being essentially unknown outside of Germany. Had Baerwald consider that as a possibility? Thus, the period from 1941 to 1983 was indeed regarded by the audio world as the "*Baerwald alignment*" era, as Löfgren's article and its results were all but unknown globally during that period.

A world war with Nazi-Germany was running at the time Baerwald's article was published (Baerwald had already migrated to the USA), so consider the possibility that the global political situation in 1941 swayed Baerwald not to acknowledge Löfgren's article published in German in a German journal. *However, a very large period of opportunity existed between the end of the war in 1945 and Baerwald's death in 1987 -* ***42 years in fact*** *- for Baerwald to issue a statement which finally confirms the true historical facts and sets the record straight between himself and Löfgren. He never did.*

By his ongoing commitment to silence, Baerwald continued, through the medium of his 1941 article, to falsely claim ownership of the development of the optimum alignment equations, right up to his death in 1987.

The writer has no doubt that Baerwald was intellectually capable of developing the mathematical results produced by Löfgren, but the simple fact is that he failed to *publicly* acknowledge that Löfgren had preceded him by three years!

However, a further injustice was done to Löfgren. To the best of the writer's knowledge, Baerwald never relinquished or denied the ongoing professional recognition and admiration accorded him by his peers and the audio world for the development of the optimum alignment solution. He chose to remain silent on this for the next 46 years, right up to his death.

Did Baerwald rationalise that the audio world was never likely to become aware of Löfgren's 1938 article, and so decided that silence was golden? His subterfuge, if that's what it truly was, remained safely hidden until its exposure in 1983 by this writer after a copy of Löfgren's article was obtained.

The Historical Facts

In summary, the writer deeply regrets not communicating with Löfgren and Baerwald in 1983 to apprise them of the historical facts which had been identified:

1. Löfgren, and not Baerwald, was the first to present an analytic treatment of lateral tracking error distortion, its origin and its effects.

2. Löfgren, and not Baerwald, was the first to show that lateral tracking error distortion is proportional to the lateral tracking error and inversely proportional to the groove radius.

3. Löfgren, and not Baerwald, was the first to develop and derive the optimum alignment formulas which minimised lateral tracking error distortion.

4. Baerwald had published seven reformulated alignment equations of Löfgren's without any attribution to Löfgren for their origin, thereby claiming them as his own work.

5. Löfgren died in 1986, and to the best of the writer's knowledge, had never publicly raised the professional injustices done to him by Baerwald.

6. Baerwald died in 1987, and to the best of the writer's knowledge, had never corrected the professional injustices he did to Löfgren by his 1941 article, and had never relinquished or denied the ongoing recognition bestowed upon him during the following 46 years – recognition which clearly and rightly belonged to Löfgren.

A Sad Summary

In the context of tonearm *alignment*, the term '*Baerwald alignment'* should be expunged from *all* books, publications and articles, and in *all* communications, including internet-based special interest groups and forums and personal communications, and replaced with the term '*Löfgren alignment'*. Why? The term 'Baerwald alignment' carries no more status than 'Bauer alignment', 'Seagrave alignment', 'Stevenson alignment' or 'Kessler and Pisha' alignment (with no disrespect intended to those authors), but most importantly, the term 'Baerwald alignment' obfuscates the true originator of the tonearm alignment formulas we have been using since 1938, or at least since 1941, which were not originated by Baerwald at all. They were originated by Löfgren. Let's keep the record straight!

GRAEME F. DENNES

Baerwald unashamedly held onto the "ownership" of tonearm alignment right up to his death in 1987. He neither relinquished nor denied the recognition accorded him after his tonearm article was published in 1941, and not even after this writer exposed the truths [7] in 1983. Baerwald's continued obfuscation of the truth was not the result of considerations such as distances, circumstances or times. In fact, his 1941 article refers to Löfgren's article three times. Baerwald exercised "ownership" of tonearm alignment from 1941 to 1987 – *a period of 46 years*.

If Baerwald truly was an honourable professional, he would have ensured the perceptions and the realities of the shortcomings expressed here did not occur. He knew what was expected of him as an acclaimed, accomplished and published professional. His standing was exemplary amongst his peers.

Finally, why did Baerwald offer the alignment formulas at his EQN (16) as being original to himself (due to the absence of anything to the contrary), knowing that Löfgren had developed the *identical* formulas three years earlier, yet he failed to formally acknowledge Löfgren role in the formula development?

Let us assume that Baerwald developed the formulas independently of Löfgren. However, once Baerwald read and understood the content of Löfgren's article mathematically, *as he did*, how could he then not say anything about parallel development? It would have been totally acceptable for two great minds to develop the identical results independently. Such has happened before. However, it appears that Baerwald clearly *and obviously* made the choice to not acknowledge Löfgren's parallel development, *if that's what it was*, of the identical solution. *Baerwald presented his solution as a new solution, originated by himself, while knowing full well that Löfgren had produced, and published, the identical solution three years earlier.* Despite that, Baerwald maintained his silence. Seems he was pipped at the post! If it was indeed a true parallel development, then up to the point where Baerwald first gained access to Löfgren's article, Baerwald could have claimed that fact and still be able to lay claim to the shared accolades with Löfgren, but we know he chose to continue to remain silent on the matter, giving no recognition to Löfgren for the part Löfgren played in tonearm alignment history.

From the foregoing, the writer has been unable to find a satisfactory explanation for Baerwald's conduct in not recognising Löfgren's results. Therefore, the writer is left to conclude, at best, that Baerwald realised he wasn't going to be the first across the finish line, so he chose not to share the limelight. In fact, he chose to *own* the limelight, of which he certainly did for the remainder of his life! That's the most disturbing aspect.

Regrettably, for Löfgren, Baerwald never addressed these issues right from the very beginning, and unapologetically left them unanswered up to his death, 46 years later.

The final line? Since 1983, the writer has continued to seek and understand the truth behind these perceptions and realities about Baerwald and his article and has found no plausible evidence or circumstances to confirm that Baerwald was acting professionally and honourably in his dealings with Löfgren, Löfgren's work and the public. This is the only conclusion left for the writer to draw from the foregoing *realities and facts*. If the readers have *facts* (not views or opinions) which rebut these conclusion, the writer would be most grateful to learn of those facts. The goal? To know the truth.

Please note:
Baerwald was an exceptional engineer and mathematician, and nothing here is intended to diminish or discredit his extensive, thorough and rigorous analysis of the origin and effects of tracking distortion, nor of his extensive analysis of the effects of errors or compromises in the mounting of tonearms. His work is exemplary.

GRAEME F. DENNES

ANALYSIS OF LÖFGREN'S EQUATIONS

NOTATION USED:

ϕ_0 = angle complementary to offset angle
β = offset angle (in this analysis)
g = overhang
R = effective arm length
D = mounting distance
r_1 = inner recorded groove radius
r_2 = outer recorded groove radius
δ = tracking error
r = any radius
p = linear offset
$= R \sin\beta$

INTRODUCTION TO LÖFGREN'S PAPER

'Löfgren A' Solution

Löfgren was the first to show that the distortion resulting from lateral tracking error was not only proportional to the tracking error but was also inversely proportional to the groove radius. This is termed the weighted tracking error (WTE).

Löfgren's primary goal was to develop equations for the optimum offset angle and overhang, which, when employed in the setting up of a tonearm and cartridge, would minimise the WTE (and tracking distortion) by minimising the maximums of the three WTE peaks across the record playing surface.

I refer to this as the 'Löfgren A' Solution.

'Löfgren B' Solution

For situations where the offset angle is fixed and non-optimum, Löfgren suggested an approach which minimises the distortion integral (LMS) across the playing surface. The 'Löfgren B' alignment calculates the appropriate overhang to achieve this.

'Löfgren C' Solution

Löfgren also considered a third alignment approach. It could be argued that the three WTE peaks are not of equal importance, and that more importance should be placed on the central peak (the peak between the null radii), where the WTE changes slowly with time.

Löfgren chose to accommodate this by minimising the distortion integral (LMS) across the playing surface, which has the effect of lowering the central WTE peak with respect to the other two peaks. This required a change to the offset angle and overhang. Löfgren did not provide a solution for the required offset angle and overhang, although this may be done today using the Solver tool from Microsoft Excel.

'LÖFGREN A' SOLUTION

A. OFFSET ANGLE

From Löfgren's EQN(41):

$$\cos \phi_0 = \frac{\rho}{R} \qquad (1)$$

As ϕ_0 and β are complementary, then

$$\sin \beta = \frac{\rho}{R} \qquad (2)$$

where $\rho = \frac{a^2}{r^*}$ from Löfgren's EQN (42) (3)

and $$a^2 = \frac{8(r_1 r_2)^2}{4 r_1 r_2 + (r_1 + r_2)^2} \qquad (4)$$

from Löfgren's EQN (40)

and $$r^* = \frac{2 r_1 r_2}{r_1 + r_2} \qquad (5)$$

from Löfgren's EQN (37)

ρ = linear offset
a^2 = product of null radii
r^* = radius at minimum of WTE function.

From (3):
$$p = \frac{a^2}{r^*}$$
$$= \frac{8(r_1 r_2)^2}{4r_1r_2 + (r_1+r_2)^2} \Big/ \frac{2r_1r_2}{r_1+r_2}$$
$$= \frac{2r_1^2r_2^2}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \cdot \frac{r_1+r_2}{2r_1r_2}$$
$$= \frac{r_1r_2(r_1+r_2)}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \qquad (6)$$

From (2):
$$\sin\beta = \frac{p}{R}$$
$$\therefore \quad \sin\beta = \frac{r_1r_2(r_1+r_2)}{R\left[\frac{1}{4}(r_1+r_2)^2 + r_1r_2\right]} \qquad (7)$$

<u>NOTE</u>: From Löfgrens FIG. 1b,
$$p = R\sin\beta \qquad (8)$$
$$= \frac{r_1r_2(r_1+r_2)}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \qquad (9)$$

B. OVERHANG

a. Exact:

$$D = \left[R^2 - a^2\right]^{1/2} \qquad (10)$$

From Löfgren's EQN (39).

But $D = R - g$

$$\therefore\ g = R - D$$

$$= R - \left[R^2 - a^2\right]^{1/2} \qquad (11)$$

From (4):

$$g = R - \left[R^2 - \frac{2 r_1^2 r_2^2}{\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2}\right]^{1/2} \qquad (12)$$

b. Approximate:

$$g \approx \frac{a^2}{2R} \qquad (13)$$

From Löfgren's EQN (47a)

$$\therefore\ g \approx \frac{r_1^2 r_2^2}{R\left[\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2\right]} \qquad (14)$$

C. ZERO TRACKING ERROR RADII

Löfgren does not provide equations for the null radii explicitely, but the product of the null radii is given by the a^2 term.

D. RADIUS OF MAXIMUM W.T.E BETWEEN NULL RADII

1. $$r^* = \frac{2r_1r_2}{r_1+r_2} \qquad (15)$$

2. $$r^* = \frac{R^2 - D^2}{R\cos\phi_0} \qquad (16)$$

$$\equiv \frac{R^2 - D^2}{R\sin\beta} \qquad (17)$$

Löfgren's EQN's (37) and (36) respectively.

E. EFFECT OF ALIGNMENT ERRORS

F. CALCULATION OF ONE PARAMETER WHEN OTHER IS FIXED.

Because of translation difficulties, no comment can be made on E. and F. above.

G. MAXIMUM WEIGHTED TRACKING ERROR

$$\left|\frac{\delta}{r}\right|_{max} = \frac{\frac{p}{r^*} - 1}{2\left[R^2 - p^2\right]^{1/2}} \qquad (18)$$

From Löfgren's EQN (43)

'LÖFGREN B' SOLUTION

A. OFFSET ANGLE

The 'Löfgren B' solution is used when the offset angle is fixed and non-optimum. Thus the offset angle is an input into the 'Löfgren B' solution, not an output.

B. OVERHANG

a. Exact:

From (11), $g = R - [R^2 - a^2]$ (20)

$= \text{Löfgren (47)}$

where $a^2 = \dfrac{3 r_1 r_2 [\rho(r_1 + r_2) - r_1 r_2]}{(r_1 + r^2)^2 - r_1 r_2}$ (21)

$= \text{Löfgren (48)}$

where $\rho = R \sin \beta$

and β = fixed, non-optimum offset angle

b. Approximate:

From (13), $g \approx \dfrac{a^2}{2R}$ (22)

$= \text{Löfgren (47a)}$

where a^2 = (21) above

and $\rho = \dfrac{r_1 r_2 (r_1 + r_2)}{\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2}$ (23)

$= \text{Löfgren (42)}$

C. ZERO TRACKING ERROR RADII

Löfgren does not provide equations for the null radii explicitely, but the product of the null radii is given by the 'a^2' term.

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

$$r^* = \frac{R^2 - D^2}{R \cos \phi_o} \qquad (24)$$

$$= \frac{R^2 - D^2}{R \sin \beta} \qquad (25)$$

From Löfgren's EQN (36).

H. SUMMARY OF LÖFGREN'S EQUATIONS

a. 'LÖFGREN A':

1. OPTIMUM OFFSET ANGLE :

$$\sin\beta = \frac{r_1 r_2 (r_1 + r_2)}{R\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (26)$$

2. OPTIMUM OVERHANG – Exact :

$$g = R - \left[R^2 - \frac{2 r_1^2 r_2^2}{\frac{1}{4}(r_1+r_2)^2 + r_1 r_2}\right]^{1/2} \qquad (27)$$

3. OPTIMUM OVERHANG – Approximate :

$$g \approx \frac{r_1^2 r_2^2}{R\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (28)$$

4. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

a. $$r^* = \frac{2 r_1 r_2}{r_1 + r_2} \qquad (29)$$

b. $$r^* = \frac{R^2 - D^2}{R \sin\beta} \qquad (30)$$

GRAEME F. DENNES

5. MAXIMUM WEIGHTED TRACKING ERROR

$$\left|\frac{\delta}{R}\right|_{max} = \frac{\frac{P}{r^*} - 1}{2\left[R^2 - p^2\right]^{1/2}} \tag{31}$$

where δ is in radians.

6. LINEAR OFFSET

$$p = R \sin\beta \tag{32}$$

$$p = \frac{r_1 r_2 (r_1 + r_2)}{\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2} \tag{33}$$

7. RADIUS OF MAXIMUM TRACKING ERROR BETWEEN NULL RADII

$$a = \sqrt{a^2} \tag{34}$$

$$= \left[\frac{2(R_1 R_2)^2}{\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2}\right]^{\frac{1}{2}} \tag{35}$$

8. MOUNTING DISTANCE

$$D = R - g \tag{36}$$

b. 'LÖFGREN B':

1. OFFSET ANGLE

Is an input to the "Löfgren B" alignment.

2. OPTIMUM OVERHANG

a. Exact: $$g = R - \left[R^2 - a^2\right]^{1/2} \qquad (37)$$

b. Approx.: $$g = \frac{a^2}{2R} \qquad (38)$$

where $$a^2 = \frac{3r_1r_2\left[p(r_1+r_2) - r_1r_2\right]}{(r_1+r_2)^2 - r_1r_2} \qquad (39)$$

where $$p = \frac{r_1r_2(r_1+r_2)}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \qquad (40)$$

and $$p = R\sin\beta \qquad (41)$$

3. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

$$r^* = \frac{R^2 - D^2}{R\sin\beta} \qquad (42)$$

4. RADIUS OF MAXIMUM TRACKING ERROR BETWEEN NULL RADII

$$a = \left[a^2\right]^{\frac{1}{2}} \tag{43}$$

$$= \left[\frac{3r_1r_2\left[p(r_1+r_2) - r_1r_2\right]}{(r_1+r_2)^2 - r_1r_2}\right]^{\frac{1}{2}} \tag{44}$$

where $$p = \frac{r_1r_2(r_1+r_2)}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \tag{45}$$

$$= R \sin\beta \tag{46}$$

5. LINEAR OFFSET

$$p = R\sin\beta \tag{47}$$

$$= \frac{r_1r_2(r_1+r_2)}{\frac{1}{4}(r_1+r_2)^2 + r_1r_2} \tag{48}$$

6. MOUNTING DISTANCE

$$D = R - g \tag{49}$$

ANALYSIS OF BAERWALD'S EQUATIONS

NOTATION USED:

α = optimum offset angle
$-d$ = optimum overhang
L = effective arm length
r_1 = inner groove radius
r_2 = outer groove radius

OTHERS USED:

γ = tracking angle
η = tracking error
$= \gamma - \alpha$
r = radius

$$\rho = \frac{1}{2}\left[a^{\frac{1}{2}} + a^{-\frac{1}{2}}\right] \qquad (1)$$

$$a = \frac{r_2}{r_1} \qquad (2)$$

$$x_m = \frac{r_m}{L} \qquad (3)$$

$$r_m = (r_1 r_2)^{\frac{1}{2}} = \text{mean groove radius} \qquad (4)$$

$$\delta = \frac{d}{L} \qquad (5)$$

$$x = \frac{r}{L}$$

η' = WTE referred to r_m

$$= \eta \cdot \frac{r_m}{r}$$

$$\eta = \eta' \cdot \frac{r}{r_m} = \text{tracking error}$$

$$\text{WTE} = \frac{\eta}{r} = \frac{\eta'}{r_m}$$

A. OFFSET ANGLE

$$\sin\alpha = \frac{2 p x_m}{p^2 + 1} \qquad (6)$$

Shown at EQN (16a) of the Reference.

Expanded Form :

a. From EQN (1) :

$$p = \tfrac{1}{2}\left[a^{1/2} + \frac{1}{a^{1/2}}\right]$$

$$= \tfrac{1}{2}\left[\frac{r_2^{1/2}}{r_1^{1/2}} + \frac{r_1^{1/2}}{r_2^{1/2}}\right]$$

$$= \tfrac{1}{2}\left[\frac{r_1 + r_2}{(r_1 r_2)^{1/2}}\right] \qquad (7)$$

$$\therefore \quad p^2 = \tfrac{1}{4}\left[\frac{(r_1 + r_2)^2}{r_1 r_2}\right] \qquad (8)$$

b. From EQN (3) :

$$x_m = \frac{r_m}{L}$$

$$= \frac{(r_1 r_2)^{1/2}}{L} \qquad (9)$$

$$\therefore \quad x_m^2 = \frac{r_1 r_2}{L^2} \qquad (10)$$

Then :

$$\sin\alpha = \frac{2 p x_m}{p^2 + 1}$$

$$= \frac{2 \cdot \frac{1}{2}\left[\frac{r_1+r_2}{(r_1 r_2)^{1/2}}\right] \cdot \frac{(r_1 r_2)^{1/2}}{L}}{\frac{1}{4}\left[\frac{(r_1+r_2)^2}{r_1 r_2}\right] + 1}$$

$$= \frac{\frac{r_1+r_2}{(r_1 r_2)^{1/2}} \cdot \frac{(r_1 r_2)^{1/2}}{L}}{\frac{1}{4}\left[\frac{(r_1+r_2)^2}{r_1 r_2}\right] + 1}$$

$$= \frac{\frac{r_1+r_2}{L} \cdot r_1 r_2}{\frac{1}{4}\left[(r_1+r_2)^2\right] + r_1 r_2}$$

$$= \frac{r_1 r_2 \, (r_1 + r_2)}{L\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (11)$$

B. OVERHANG

$$-\delta = 1 - \left[1 - \frac{2 x_m^2}{p^2 + 1} \right]^{1/2} \qquad (12)$$

Shown at EQN (16b) of the Reference.

Expanded Form

a. From EQN (8) :

$$p^2 = \frac{1}{4} \left[\frac{(r_1 + r_2)^2}{r_1 r_2} \right]$$

b. From EQN (10) :

$$x_m^2 = \frac{r_1 r_2}{L^2}$$

Then :

$$-\delta = 1 - \left[1 - \frac{2x_m^2}{p^2+1} \right]^{1/2}$$

$$= 1 - \left[1 - \frac{2 \cdot \frac{r_1 r_2}{L^2}}{\frac{1}{4}\left[\frac{(r_1+r_2)^2}{r_1 r_2}\right] + 1} \right]^{1/2}$$

$$= 1 - \left[1 - \frac{2 \cdot \frac{r_1 r_2}{L^2} \cdot r_1 r_2}{\frac{1}{4}\left[\frac{(r_1+r_2)^2}{r_1 r_2}\right] + r_1 r_2} \right]^{1/2}$$

$$= 1 - \left[1 - \frac{2 \cdot \frac{r_1^2 r_2^2}{L^2}}{\frac{1}{4}(r_1+r_2)^2 + r_1 r_2} \right]^{1/2} \qquad (13)$$

$$\therefore \; -d = L - \left[L^2 - \frac{2 r_1^2 r_2^2}{\frac{1}{4}(r_1+r_2)^2 + r_1 r_2} \right]^{1/2} \qquad (14)$$

APPROXIMATION OF OVERHANG.

Baerwald also provided an approximation for the optimum overhang, as well as the exact solution previously expanded.
The approximation is given by two expressions, the second simply being a partial expansion of the first.

$$-\delta \doteq \frac{x_m^2}{p^2+1} \qquad (15)$$

$$-d \doteq \frac{r_1 r_2}{L\left[\dfrac{\left(\dfrac{r_1+r_2}{2}\right)^2}{r_1 r_2}+1\right]} \qquad (16)$$

Both shown at EQN (16b) of the Reference.

EQN (15) above can be expanded as follows:

$$\frac{x_m^2}{p^2+1} = \frac{\dfrac{r_1 r_2}{L^2}}{\dfrac{1}{4}\left[\dfrac{(r_1+r_2)^2}{r_1 r_2}\right]+1}$$

$$= \frac{\dfrac{r_1^2 r_2^2}{L^2}}{\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2}$$

$$= \frac{r_1^2 r_2^2}{L^2\left[\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2\right]} \qquad (17)$$

$$\therefore \quad -\delta \doteq \frac{r_1^2 r_2^2}{L^2\left[\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2\right]} \qquad (18)$$

$$\therefore \quad -d \doteq \frac{r_1^2 r_2^2}{L\left[\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2\right]} \qquad (19)$$

C. ZERO TRACKING ERROR RADII.

a. $$r = \frac{2r_1r_2}{\left(1+\frac{1}{\sqrt{2}}\right)r_2 + \left(1-\frac{1}{\sqrt{2}}\right)r_1} \qquad (20)$$

b. $$r = \frac{2r_1r_2}{\left(1-\frac{1}{\sqrt{2}}\right)r_2 + \left(1+\frac{1}{\sqrt{2}}\right)r_1} \qquad (21)$$

Shown at EQN (16e) of the Reference, and provide, respectively, the inner and outer zero tracking ~~errot~~ radii.

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

$$r_0 = \frac{2 r_1 r_2}{r_1 + r_2} \qquad (22)$$

Given as part of Baerwald's EQN (16e).

E. EFFECTS OF ALIGNMENT ERRORS

This is discussed on pages 612-618 of the Reference, and includes some very useful graphs.

F. CALCULATION OF ONE PARAMETER, WHEN THE OTHER IS FIXED

Referring to Baerwald's paper:

Although not given explicitely, the optimum overhang for a given offset angle can be calculated using EQN (21), (22) and (23). Calculation of the optimum offset angle for a given overhang is discussed on pages 617 and 618.

G. SUMMARY OF BAERWALD'S EQUATIONS

a. OPTIMUM OFFSET ANGLE.

$$\sin\alpha = \frac{r_1 r_2 (r_1 + r_2)}{L\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (23)$$

b. OPTIMUM OVERHANG (EXACT)

$$-d = L - \left[L^2 - \frac{2 r_1^2 r_2^2}{\frac{1}{4}(r_1+r_2)^2 + r_1 r_2}\right]^{1/2} \qquad (24)$$

c. OPTIMUM OVERHANG (APPROXIMATE)

$$-d \doteq \frac{r_1^2 r_2^2}{L\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (25)$$

d. ZERO ERROR RADII

$$r = \frac{2 r_1 r_2}{\left(1 \pm \frac{1}{\sqrt{2}}\right) r_2 + \left(1 \mp \frac{1}{\sqrt{2}}\right) r_1} \qquad (26)$$

e. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

$$r_0 = \frac{2 r_1 r_2}{r_1 + r_2} \qquad (27)$$

CORRECTIONS TO BAERWALD'S PAPER

1. On page 606 of Baerwald's paper, at the last sentence on the page, Baerwald states: "... it seems that the Chebyshev manner is somewhat preferable to the minimal mean square [method] suggested by Löfgren.". (This comment is made in relation to a comparison between his (Baerwald's) solution and Löfgren's ['Löfgren A'] solution.)

   In fact, Löfgren does not refer to the minimal mean square method (method of least squares) in relation to the 'Löfgren A' solution. Löfgren refers to it in relation to the 'Löfgren C' solution, during its discussion at the last paragraph on page 359 of his paper.

2. At EQN (20) on page 613 of Baerwald's paper, the second equation, with the numerator of r_1, is incorrect.
   In the denominator, the square brackets term is shown as being squared. It should not be squared. The inner brackets term ($1 - r_1/r_2$) should be squared.

ANALYSIS OF BAUER'S EQUATIONS

NOTATION USED :

β = optimum offset angle in radians
D = optimal overhang
l = effective arm length
r_1 = inner recorded groove radius
r_2 = outer recorded groove radius

A. OFFSET ANGLE IN RADIANS

$$\beta = \frac{r_1(1 + r_1/r_2)}{\ell\left[\frac{1}{4}(1 + r_1/r_2)^2 + r_1/r_2\right]} \quad (1)$$

From EQN (29) on page 115 of his article.

Expanded Form :

From EQN (1) :

$$\beta = \frac{r_1 + r_1^2/r_2}{\ell\left[\frac{1}{4}(1 + 2r_1/r_2 + r_1^2/r_2^2) + r_1/r_2\right]}$$

$$= \frac{r_1 r_2^2 + r_1^2 r_2}{\ell\left[\frac{1}{4}(r_1^2 + 2r_1 r_2 + r_2^2) + r_1 r_2\right]}$$

$$= \frac{r_1 r_2(r_1 + r_2)}{\ell\left[\frac{1}{4}(r_1 + r_2)^2 + r_1 r_2\right]} \quad (2)$$

B. OVERHANG

$$D = \frac{r_1^2}{\ell\left[\frac{1}{4}(1 + r_1/r_2)^2 + r_1/r_2\right]} \qquad (3)$$

From EQN (28) on page 115 of his article.

Expanded Form :

From EQN (3) :

$$D = \frac{r_1^2}{\ell\left[\frac{1}{4}(1 + 2r_1/r_2 + r_1^2/r_2^2) + r_1/r_2\right]}$$

$$= \frac{r_1^2 r_2^2}{\ell\left[\frac{1}{4}(r_1^2 + 2r_1r_2 + r_2^2) + r_1r_2\right]}$$

$$= \frac{r_1^2 r_2^2}{\ell\left[\frac{1}{4}(r_1 + r_2)^2 + r_1r_2\right]} \qquad (4)$$

EQN (4) is identical to the approximation given by Löfgren.

GRAEME F. DENNES

C. ZERO TRACKING ERROR RADII

Bauer does not discuss this.

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

Bauer does not discuss this.

E. EFFECTS OF ALIGNMENT ERRORS

Bauer does not discuss this.

F. CALCULATION OF ONE PARAMETER, WHEN THE OTHER IS FIXED.

Bauer discusses this on pages 114 and 115 of the Reference, and is also the subject of Reference 14, which uses a direct calculation method.

G. SUMMARY OF BAUER'S EQUATIONS

a. OPTIMUM OFFSET ANGLE

$$\beta = \frac{r_1 r_2 (r_1 + r_2)}{\ell\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \quad \text{(radians)} \qquad (5)$$

b. OPTIMUM OVERHANG

$$D = \frac{r_1^2 r_2^2}{\ell\left[\frac{1}{4}(r_1+r_2)^2 + r_1 r_2\right]} \qquad (6)$$

NOTE

As Bauer made two simplifying assumptions in his paper, the use of his equations for optimum offset angle and overhang will result in a slightly higher distortion being experienced compared to Löfgren's equations.

The assumptions were:

1. $D^2 \ll 2\ell D$, so neglect D^2
2. $\sin\phi = \phi$ in radians

GRAEME F. DENNES

ANALYSIS OF SEAGRAVE'S EQUATIONS

NOTATION USED :

β = optimum offset angle
D = optimum overhang
L = effective arm length
R_1 = inner recorded groove radius
R_2 = outer recorded groove radius

A. OFFSET ANGLE

$$\sin \beta = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2\right]} \qquad (1)$$

From EQN (22) on page 25 of Part 2.

B. OVERHANG

Seagrave gave two methods for the solution of optimum overhang.

1. Using his EQN (15) and EQN (23) and solving for D. This solution is identical to Löfgren's exact solution for optimum overhang.

2. Using his EQN (17) and EQN (23), to produce an approximation for overhang.

Seagrave's EQN (23) is identical to the optimum overhang equation given by Bauer. Although the second method above is also an approximation, it is a much better one than Bauer's solution.

1. EXACT SOLUTION

$$D_1 = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \qquad (2)$$

From EQN (23) on page 25 of Part 2.

where $$D_1 = D\left[1 - \tfrac{1}{2} D/L\right] \qquad (3)$$

From EQN (15) on page 25 of Part 2.

From EQN (3) : $$D_1 = D - \frac{1}{2}\frac{D^2}{L}$$

$$\therefore \frac{D^2}{2L} - D + D_1 = 0$$

$$\therefore D^2 - 2LD + 2LD_1 = 0 \qquad (4)$$

Solving EQN (4) as a quadratic :

$$D = \frac{2L \pm \left[4L^2 - 8LD_1\right]^{1/2}}{2} \qquad (5)$$

$$= L \pm \left[\frac{4L^2 - 8LD_1}{4}\right]^{1/2}$$

$$= L \pm \left[L^2 - 2LD_1\right]^{1/2}$$

For a real solution, use the negative sign.

$$\therefore D = L - \left[L^2 - 2LD_1\right]^{1/2} \qquad (6)$$

Subs. EQN (2) into EQN (6) :

GRAEME F. DENNES

$$D = L - \left[L^2 - \frac{2LR_1^2R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \right]^{1/2}$$

$$\therefore \quad D = L - \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \right]^{1/2} \qquad (7)$$

2. APPROXIMATE SOLUTION

$$D \doteq D_1\left[1 + \frac{1}{2}\frac{D_1}{L}\right] \qquad (8)$$

From EQN (17) on page 25 of Part 2.

$$\text{where } D_1 = \frac{R_1^2R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \qquad (9)$$

From EQN (23) on page 25 of Part 2.

Although this method is an approximation, it produces a more accurate solution than Bauer's solution, which was also an approximation.

C. ZERO TRACKING ERROR RADII

Not discussed

D. RADIUS OF MAXIMUM WEIGHTED TRACKING ERROR BETWEEN NULL RADII.

$$R = \frac{2 D_1}{\sin\beta} \qquad (10)$$

From text on page 25 of Part II.

$$\therefore \quad R = 2 . D\left[1 - \frac{D}{2L}\right] / \sin\beta \qquad (11)$$

using Seagrave's EQN (15).

$$= \left[2D - \frac{D^2}{L}\right] / \sin\beta \qquad (12)$$

$$= \frac{2LD - D^2}{L \sin\beta} \qquad (13)$$

E. EFFECT OF ALIGNMENT ERRORS

This is discussed on pages 26 and 27 of Part II.

GRAEME F. DENNES

F. <u>OPTIMUM GEOMETRY WHEN ONE PARAMETER IS FIXED.</u>

Discussed on pages 25 and 26 of Part II, and uses a direct calculation method.

G. SUMMARY OF SEAGRAVE'S EQUATIONS

a. OPTIMUM OFFSET ANGLE

$$\sin\beta = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2\right]} \qquad (14)$$

b. OPTIMUM OVERHANG

Exact (15)

$$D = L - \left[L^2 - \frac{2 R_1^2 R_2^2}{\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2}\right]^{1/2}$$

Approximate

$$D \doteq D_1\left[1 + \frac{1}{2}\frac{D_1}{L}\right] \qquad (16)$$

$$\text{where } D_1 = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2\right]} \qquad (17)$$

c. RADIUS OF MAXIMUM WEIGHTED TRACKING ERROR BETWEEN NULL RADII

$$R = \frac{2LD - D^2}{L \sin\beta} \qquad (18)$$

D and β do not have to be optimum, as this equation is derived directly from the tracking equation.

CORRECTIONS TO SEAGRAVE'S PAPER

1. EQN (16) on page 25 of Part 2 reads: $\sin\phi = \tfrac{1}{2}(R/L) + D_1/L$

   It should read: $\sin\phi = \tfrac{1}{2}(R/L) + D_1/R$

2. Due to a production problem in the printing of Part 2, EQN (27c) is incorrect. The correct EQN (27c) will now be derived.

   From EQN (18) in Part 2, we require m2 = -m1, ie

$$\frac{1}{2L} + \frac{D1}{R_2^2} - \frac{\sin\beta}{R_2} = -\left[\frac{1}{2L} + \frac{D1}{R_1^2} - \frac{\sin\beta}{R_1}\right]$$

$$= -\frac{1}{2L} - \frac{D1}{R_1^2} + \frac{\sin\beta}{R_1}$$

$$\text{ie}\quad \frac{D1}{R_1^2} + \frac{D1}{R_2^2} = \frac{\sin\beta}{R_1} + \frac{\sin\beta}{R_2} - \frac{1}{2L} - \frac{1}{2L}$$

$$\text{ie}\quad D1\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) = \sin\beta\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{1}{L}$$

$$\text{so}\quad D1 = \frac{\sin\beta\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{1}{L}}{\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)} \qquad (27c)$$

ANALYSIS OF STEVENSON'S EQUATIONS

NOTATION USED :

θ_D = optimum offset angle
f = optimum overhang
l = effective arm length
x_2 = outer recorded groove radius
x_3 = inner recorded groove radius
x_0' = Outer zero tracking error radius
x_0 = inner zero tracking error radius
x_p = maximum weighted tracking error radius between x_0', x_0
x_m = maximum tracking error radius between x_0', x_0

INTRODUCTION TO STEVENSON'S PAPER

Stevenson considered two approaches to alignment optimisation.

Stevenson's main approach focuses on the tracking distortion at the inner groove area, and places the inner null radius at the inner groove radius, and equalises the |WTE| at R_W and R_2. Although this results in a somewhat higher WTE over most of the record playing surface, the distortion due to WTE at the innermost groove area will be minimal. I refer to this as the 'Stevenson A' Solution.

His alternative approach is the three-point, equal-WTE solution, and is shown under the 'Stevenson B' Solution.

'STEVENSON A' SOLUTION

This solution uses x_0 and x_2 as inputs, to enable the inner null radius (x_0) to be placed at the inner groove radius. The 'Stevenson A' equations are perfectly consistent with the 'Stevenson B' equations.

A. OFFSET ANGLE

$$\sin \theta_D = \frac{x_m^2}{L.x_p} \qquad (1)$$

$$x_m^2 = \frac{x_0^2 x_2}{0.8284 x_0 + 0.1716 x_2} \qquad (2)$$

$$x_p = \frac{2 x_0 x_m^2}{x_0^2 + x_m^2} \qquad (3)$$

from page 218, Part 1 of the article.

Also,
$$x_m^2 = \frac{x_0^2 x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2} \qquad (4)$$

When EQNs. (3) and (4) are used with EQN (1):

$$\sin \theta_D = \frac{1}{2L}\left[x_0 + \frac{x_0 x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2}\right] \qquad (5)$$

EQN (5) is simply a rearrangement of EQN (15) following to enable x_0 and x_2 as inputs.

B. <u>OVERHANG</u>

$$f = l - \left(l^2 - x_m^2\right)^{\frac{1}{2}} \qquad (6)$$

from EQN (8), page 218 of Part I of paper.

where x_m^2 = EQN (4).

Also, $x_o' = \dfrac{x_m^2}{x_o}$ (page 218, Part 1)

so $x_m^2 = x_o x_o'$

and $$f = l - \left(l^2 - x_o x_o'\right)^{\frac{1}{2}} \qquad (7)$$

$= l -$ Mounting Centre

C. <u>Remaining Variables</u>

1. $$x_3 = \frac{x_m^2}{x_o + \dfrac{x_m^2}{x_o} - \dfrac{x_m^2}{x_2}} \qquad (8)$$

2. $x_o' = \dfrac{x_m^2}{x_o}$ where x_m^2 = EQN (4) (9)

ie $$x_o' = \text{EQN}(4) \cdot \frac{1}{x_o}$$

$$= \frac{x_o x_2}{(2\sqrt{2} - 2)x_o - (2\sqrt{2} - 3)x_2} \qquad (10)$$

'STEVENSON B' SOLUTION

This uses x_2 and x_3 as inputs.

A. OFFSET ANGLE

(i) $$\sin \theta_D = \frac{x_m^2}{l.x_p} \qquad (11)$$

where $$x_m^2 = \frac{8x_2^2 x_3^2}{x_2^2 + x_3^2 + 6x_2x_3} \qquad (12)$$

and $$x_p = \frac{2x_2x_3}{x_2+x_3} \qquad (13)$$

From (11)

$$\sin \theta_D = \frac{8x_2^2x_3^2}{l(x_2^2+x_3^2+6x_2x_3).\frac{2x_2x_3}{x_2+x_3}} \qquad (14)$$

$$= \frac{8x_2^2x_3^2(x_2+x_3)}{l(x_2^2+x_3^2+6x_2x_3)2x_2x_3}$$

$$= \frac{4x_2^2x_3^2(x_2+x_3)}{l(x_2^2+x_3^2+6x_2x_3)x_2x_3}$$

$$= \frac{4x_2x_3(x_2+x_3)}{l(x_2^2+x_3^2+6x_2x_3)}$$

$$\sin\theta_D = \frac{4x_2x_3(x_2+x_3)}{l\left[(x_2+x_3)^2+4x_2x_3\right]}$$

ie $$\sin\theta_D = \frac{x_2x_3(x_2+x_3)}{l\left[\frac{1}{4}(x_2+x_3)^2+x_2x_3\right]} \qquad (15)$$

(ii) $$\sin\theta_D = \frac{1}{2L}\left(x_o+\frac{x_m^2}{x_o}\right) \qquad (16)$$

from EQN. (7), page 218 of Part 1.

Now, $x_o' = \frac{x_m^2}{x_o}$ (from further down page)

Substituting into equation above:

$$\sin\theta_D = \frac{1}{2L}\left(x_o+x_o'\right)$$

$$= \frac{x_o+x_o'}{2L} \qquad (17)$$

B. <u>OVERHANG</u>

(i) From EQN (8) on page 218, Part I:

$$f = l - \left(l^2 - x_m^2\right)^{\frac{1}{2}} \qquad (18)$$

$$= l - \left[l^2 - \frac{8x_2^2x_3^2}{x_2^2 + x_3^2 + 6x_2x_3}\right]^{1/2} \qquad (19)$$

$$= l - \left[l^2 - \frac{8x_2^2x_3^2}{(x_2+x_3)^2 + 4x_2x_3}\right]^{1/2}$$

$$= l - \left[l^2 - \frac{2x_2^2x_3^2}{\frac{1}{4}(x_2+x_3)^2 + x_2x_3}\right]^{1/2} \qquad (20)$$

(ii) Also using EQN (8) on page 218 of Part I:

$$f = l - \left(l^2 - x_m^2\right)^{\frac{1}{2}}$$

Substituting $x_m^2 = x_0 x_0'$ (from before),

ie $$f = l - \left(l^2 - x_0 x_0'\right)^{\frac{1}{2}} \qquad (21)$$

C. ZERO TRACKING ERROR RADII

$$x_0 = \frac{x_2 x_3}{0.8536 x_2 + 0.1464 x_3} \qquad (22)$$

$$x_0' = \frac{x_2 x_3}{0.1464 x_2 + 0.8536 x_3} \qquad (23)$$

From page 218 of the article.

From EQN (22):

$$x_0 = \frac{x_2 x_3}{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) x_2 + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) x_3} \qquad (24)$$

$$= \frac{x_2 x_3}{\frac{1}{2}\left[\left(1 + \frac{1}{\sqrt{2}}\right) x_2 + \left(1 - \frac{1}{\sqrt{2}}\right) x_3\right]}$$

$$= \frac{2 x_2 x_3}{\left(1 + \frac{1}{\sqrt{2}}\right) x_2 + \left(1 - \frac{1}{\sqrt{2}}\right) x_3} \qquad (25)$$

From EQN (23) :

$$x_0' = \frac{x_2 x_3}{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)x_2 + \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)x_3} \tag{26}$$

$$= \frac{x_2 x_3}{\frac{1}{2}\left[\left(1 - \frac{1}{\sqrt{2}}\right)x_2 + \left(1 + \frac{1}{\sqrt{2}}\right)x_3\right]}$$

$$= \frac{2 x_2 x_3}{\left(1 - \frac{1}{\sqrt{2}}\right)x_2 + \left(1 + \frac{1}{\sqrt{2}}\right)x_3} \tag{27}$$

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

1. $$x_p = \frac{2x_2x_3}{x_2+x_3} \tag{28}$$

2. From right column, page 218 of Part I:

$$x_p = \frac{2x_m^2}{x_0 + \frac{x_m^2}{x_0}} \tag{29}$$

$$= \frac{2x_0 x_m^2}{x_0^2 + x_m^2} \qquad \left(x_m^2 = x_0 x_0'\right)$$

$$= \frac{2x_0 \cdot x_0 x_0'}{x_0^2 + x_0 x_0'}$$

$$= \frac{2x_0 x_0'}{x_0 + x_0'} \tag{30}$$

E. EFFECTS OF ALIGNMENT ERRORS

Discussed on page 316 of Part 2 of article.

F. OPTIMISATION WHEN ONE PARAMETER FIXED

Discussed on page 317 of Part 2 of article, and is achieved using a trial and error procedure, and not by direct calculation.

G. SUMMARY OF STEVENSON'S EQUATIONS

1. SOLUTION A : x_0, x_2 GIVEN

a. OPTIMUM OFFSET ANGLE

$$\sin \Theta_D = \frac{x_m^2}{L \cdot x_p} \qquad (31)$$

$$\text{where } x_m^2 = \frac{x_0^2 x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2} \qquad (32)$$

$$\text{and } x_p = \frac{2x_0 x_m^2}{x_0^2 + x_m^2} \qquad (33)$$

$$\sin \Theta_D = \frac{1}{2L}\left[x_0 + \frac{x_0 x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2}\right] \qquad (34)$$

b. OPTIMUM OVERHANG

$$f = l - (l^2 - x_m^2)^{\frac{1}{2}} \qquad (35)$$

$$= l - \text{Mounting Centre} \qquad (36)$$

c. THEORETICAL INNER GROOVE RADIUS

$$x_3 = \frac{x_m^2}{x_0 + \frac{x_m^2}{x_0} - \frac{x_m^2}{x_2}} \qquad (37)$$

d. OUTER NULL RADIUS

$$x_0' = \frac{x_0 x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2} \qquad (38)$$

2. SOLUTION B: x_2, x_3 GIVEN

a. OPTIMUM OFFSET ANGLE

$$\sin\theta_D = \frac{x_2 x_3 (x_2 + x_3)}{l\left[\frac{1}{4}(x_2+x_3)^2 + x_2 x_3\right]} \qquad (39)$$

$$\sin\theta_D = \frac{x_o + x_o'}{2L} \qquad (40)$$

b. OPTIMUM OVERHANG

$$f = l - \left[l^2 - \frac{2x_2^2 x_3^2}{\frac{1}{4}(x_2+x_3)^2 + x_2 x_3}\right]^{1/2} \qquad (41)$$

$$= l - \left[l^2 - x_o x_o'\right]^{\frac{1}{2}} \qquad (42)$$

$$= l - \left[l^2 - x_m^2\right]^{\frac{1}{2}} \qquad (43)$$

c. ZERO TRACKING ERROR RADII

$$x_o, x_o' = \frac{2x_2 x_3}{\left(1 \pm \frac{1}{\sqrt{2}}\right)x_2 + \left(1 \mp \frac{1}{\sqrt{2}}\right)x_3} \qquad (44)$$

d. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

$$x_p = \frac{2x_2x_3}{x_2+x_3} \tag{45}$$

$$= \frac{2x_0x_0'}{x_0+x_0'} \tag{46}$$

e. RADIUS OF MAXIMUM TRACKING ERROR BETWEEN NULL RADII

$$x_m = \left(x_0x_0'\right)^{\frac{1}{2}} \tag{47}$$

$$= \left[\frac{x_0^2x_2}{(2\sqrt{2}-2)x_0 - (2\sqrt{2}-3)x_2}\right]^{\frac{1}{2}} \tag{48}$$

$$= \left[\frac{8x_2^2x_3^2}{x_2^2 + x_3^2 + 6x_2x_3}\right]^{\frac{1}{2}}$$

$$= \left[\frac{8(x_2x_3)^2}{(x_2+x_3)^2 + 4x_2x_3}\right]^{\frac{1}{2}} \tag{49}$$

ANALYSIS OF KESSLER AND PISHA'S EQUATIONS

NOTATION USED :

a = optimum offset angle
d = optimum overhang
L = effective arm length
R_1 = inner recorded groove radius
R_2 = outer recorded groove radius
r_1 = inner zero tracking error radius
r_2 = outer zero tracking error radius

NOTE: The authors use r_1 and r_2 for the inner and outer recorded groove radii in EQN. (1a), (1b) and (2) on page 77 of their article.

The authors then use r_1 and r_2 for the inner and outer null radii in EQN. (3) and (4) on page 77 of their article.

This analysis adjusts for this ambiguity.

A. OFFSET ANGLE

$$\sin a = \frac{R_1 + R_2}{L\left[\dfrac{\left(\dfrac{R_1+R_2}{2}\right)^2}{R_1 R_2} + 1\right]} \qquad (1)$$

The authors provide this expression at Eqn (2) on page 77 of their article, except upper case has been used for the inner and outer groove radii to prevent confusion.

Expanded Form :

From EQN (1) :

$$\sin a = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\left(\dfrac{R_1+R_2}{2}\right)^2 + R_1 R_2\right]} \qquad (2)$$

$$= \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1 R_2\right]} \qquad (3)$$

B. OVERHANG

$$d = L - M$$

$$= L - \left[\frac{r_2(L^2 + r_1^2) - r_1(L^2 + r_2^2)}{r_2 - r_1} \right]^{\frac{1}{2}} \quad (4)$$

The authors provide the expression for the Mounting Distance at EQN (3) on page 77 of their article.

Here, r_1 and r_2 are, respectively, the inner and outer null radii, and are shown at EQNs (1a) and (1b) of the article.

The equations are reproduced below, except that upper case has been used for the inner and outer groove radii to prevent confusion.

r_1 = inner zero tracking error radius

$$= \frac{2R_1R_2}{\left(1 + \frac{1}{\sqrt{2}}\right)R_2 + \left(1 - \frac{1}{\sqrt{2}}\right)R_1} \quad (5)$$

r_2 = outer zero tracking error radius

$$= \frac{2R_1R_2}{\left(1 - \frac{1}{\sqrt{2}}\right)R_2 + \left(1 + \frac{1}{\sqrt{2}}\right)R_1} \quad (6)$$

Expanded Form

From EQN (4) :

$$d = L - \left[\frac{L^2 r_2 + r_1^2 r_2 - L^2 r_1 - r_1 r_2^2}{r_2 - r_1} \right]^{1/2}$$

$$= L - \left[\frac{L^2 (r_2 - r_1) - r_1 r_2 (r_2 - r_1)}{r_2 - r_1} \right]^{1/2}$$

$$= L - \left[L^2 - r_1 r_2 \right]^{1/2} \qquad (7)$$

Now,

$$r_1 r_2 = \frac{2 R_1 R_2}{\left(1 + \frac{1}{\sqrt{2}}\right) R_2 + \left(1 - \frac{1}{\sqrt{2}}\right) R_1} \cdot \frac{2 R_1 R_2}{\left(1 - \frac{1}{\sqrt{2}}\right) R_2 + \left(1 + \frac{1}{\sqrt{2}}\right) R_1}$$

$$= \frac{4 R_1^2 R_2^2}{R_2^2/2 + \left(\frac{3}{2} + \sqrt{2}\right) R_1 R_2 + \left(\frac{3}{2} - \sqrt{2}\right) R_1 R_2 + R_1^2/2}$$

$$= \frac{4 R_1^2 R_2^2}{R_2^2/2 + \frac{3}{2} R_1 R_2 + \sqrt{2} R_1 R_2 + \frac{3}{2} R_1 R_2 - \sqrt{2} R_1 R_2 + R_1^2/2}$$

$$= \frac{4R_1^2R_2^2}{\frac{1}{2}\left[R_1^2 + R_2^2 + 6R_1R_2\right]}$$

$$= \frac{4R_1^2R_2^2}{\frac{1}{2}\left[(R_1+R_2)^2 + 4R_1R_2\right]}$$

$$= \frac{2R_1^2R_2^2}{\frac{1}{4}\left[(R_1+R_2)^2 + 4R_1R_2\right]}$$

$$= \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \qquad (8)$$

Now, substitute (8) into (7) :

$$d = L - \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2}\right]^{1/2} \qquad (9)$$

C. ZERO TRACKING ERROR RADII

$$\text{Inner null radius} = \frac{2R_1R_2}{\left(1+\frac{1}{\sqrt{2}}\right)R_2 + \left(1-\frac{1}{\sqrt{2}}\right)R_1} \qquad (10)$$

$$\text{Outer null radius} = \frac{2R_1R_2}{\left(1-\frac{1}{\sqrt{2}}\right)R_2 + \left(1+\frac{1}{\sqrt{2}}\right)R_1} \qquad (11)$$

From EQNs (1a) and (1b) on page 77 of the article, except upper case has been used for the inner and outer groove radii to prevent confusion.

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

The solution is not given in the Reference.

E. EFFECTS OF ALIGNMENT ERRORS

This is discussed on pages 82-86 of the Reference.

F. CALCULATION OF ONE PARAMETER, WHEN THE OTHER IS FIXED

This is not discussed in the Reference.

G. SUMMARY OF KESSLER & PISHA'S EQUATIONS.

a. OPTIMUM OFFSET ANGLE

$$\sin a = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1 R_2\right]} \qquad (12)$$

b. OPTIMUM OVERHANG

$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1 R_2}\right]^{1/2} \qquad (13)$$

c. ZERO TRACKING ERROR RADII

$$R = \frac{2 R_1 R_2}{\left(1 \pm \frac{1}{\sqrt{2}}\right)R_2 + \left(1 \mp \frac{1}{\sqrt{2}}\right)R_1} \qquad (14)$$

COMPARISON OF OPTIMUM EQUATIONS

This section compares the optimum equations given in the foregoing analyses of the six major tone arm geometry papers.

To simplify the comparison, the following notation will be adopted.

β = optimum offset angle
d = optimum overhang
L = effective arm length
R_1 = inner recorded groove radius
R_2 = outer recorded groove radius
R_{01} = inner zero tracking error or null radius
R_{02} = outer zero tracking error or null radius
M = mounting distance
= $L - d$

Basic Summary of the Comparison

AUTHOR	OFFSET ANGLE Exact	OFFSET ANGLE Approx.	OVERHANG Exact	OVERHANG Better	OVERHANG Approx.
LÖFGREN	•		•		•
BAERWALD	•		•		•
BAUER		•			•
SEAGRAVE	•		•	•	
STEVENSON	•		•		
KESSLER/PISHA	•		•		

A. OPTIMUM OFFSET ANGLE

a. LÖFGREN :

1. 'LÖFGREN A' : $$\sin\beta = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2\right]}$$

2. 'LÖFGREN B' : Is an input.

b. BAERWALD : As 'LÖFGREN A'

c. BAUER : β (radians) = $\sin\beta$ as above

d. SEAGRAVE : As 'LÖFGREN A'

e. STEVENSON 'STEVENSON B' :

As 'LÖFGREN A'

f. KESSLER/PISHA : As 'LÖFGREN A'

B. OPTIMUM OVERHANG

a. LÖFGREN : ('Löfgren A')

1. Exact :

$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \right]^{1/2}$$

2. Approx :

$$d = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}$$

b. LÖFGREN : ('Löfgren B')

1. Exact :

$$d = L - \left[L^2 - a^2 \right]^{1/2}$$

2. Approx :

$$d = \frac{a^2}{2L}$$

where $$a^2 = \frac{3R_1R_2\left[p(R_1+R_2) - R_1R_2\right]}{(R_1+R_2)^2 - R_1R_2}$$

and $$p = \frac{R_1R_2(R_1+R_2)}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2}$$

and $$p = L \sin\beta$$

GRAEME F. DENNES

c. BAERWALD :

1. Exact :
$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \right]^{1/2}$$

2. Approx :
$$d = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}$$

d. BAUER :

$$d = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}$$

e. SEAGRAVE :

1. Exact :
$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \right]^{1/2}$$

2. Approx :
$$d = D_1\left[1 + \frac{D_1}{2L}\right]$$

where
$$D_1 = \frac{R_1^2 R_2^2}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}$$

f. STEVENSON 'STEVENSON B':

$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1 R_2} \right]^{1/2}$$

$$= L - \left[L^2 - R_{01} R_{02} \right]^{\frac{1}{2}}$$

g. KESSLER/PISHA :

$$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1 R_2} \right]^{1/2}$$

C. ZERO TRACKING ERROR RADII

a. LÖFGREN : Not given explicitely, although their product is given by his a^2 term.

b. BAERWALD : $R = \dfrac{2R_1R_2}{\left(1 \pm \frac{1}{\sqrt{2}}\right)R_1 + \left(1 \mp \frac{1}{\sqrt{2}}\right)R_2}$

c. BAUER : Not Given

d. SEAGRAVE : Not Given

e. STEVENSON : As Baerwald's

f. KESSLER/PISHA : As Baerwald's

D. RADIUS OF MAXIMUM WEIGHTED TRACKING ERROR BETWEEN NULL RADII

a. LÖFGREN : 1. $$R = \frac{L^2 - M^2}{L \sin\beta}$$

2. $$R = \frac{2R_1R_2}{R_1 + R_2}$$

b. BAERWALD : $$R = \frac{2R_1R_2}{R_1 + R_2}$$

c. BAUER : $$R = \frac{2d}{\beta \text{ (radians)}}$$

d. SEAGRAVE : $$R = \frac{2Ld - d^2}{L \sin\beta}$$

e. STEVENSON : 1. $$R = \frac{2R_1R_2}{R_1 + R_2}$$

2. $$R = \frac{2R_{01}R_{02}}{R_{01} + R_{02}}$$

f. KESSLER/PISHA : Not Given

COMPREHENSIVE LISTING OF DESIGN EQUATIONS

Notation Used

β	Offset angle
d	Overhang
L	Effective arm length
M	Mounting distance = L - d
R_1	Inner recorded groove radius
R_2	Outer recorded groove radius
R_{01}	Inner null radius
R_{02}	Outer null radius
R_{null}	The null radii.
R_a	Radius of greatest (angular) lateral tracking error between the null radii
R_w	Radius of greatest weighted tracking error between the null radii
R	any radius
Ω	Turntable angular velocity
p	Linear offset
V	Recorded velocity
ε	Distortion
α	Lateral tracking angle
φ	Lateral tracking error = $\alpha - \beta$
γ	angle complementary to α
γ_0	γ when $\varphi = 0$ γ_0, β are complementary

Exact Alignment Formulas:

These formulas are exact (except where noted) and are independent of the alignment method used. They may be universally applied.

WTE = TAN (ASIN ((R^2 + L^2 - M^2) / (2 * L * R)) - β) / R
TA = ASIN((R^2 + L^2 – M^2) / (2 * L * R))
TE = TA – β
M = L – d
M = SQRT (L^2 – Ra^2)
L^2 – M^2 = 2 * L * d – d^2
Ra^2 = L^2 – M^2
Ra^2 = 2 * L * d – d^2
Ra^2 = R_{01} * R_{02}
d = L – SQRT (L^2 – Ra^2)
Rw = Ra^2 / p <<< Uses small angle approximations
p = L * sin β
p = (R_{01} + R_{02}) / 2
R_{01} = p –SQRT (p^2 – Ra^2)
R_{02} = p + SQRT (p^2 – Ra^2)

In the formulas above, R = (some) radius, Ra = radius at the minimum of the angular tracking error curve, Rw = radius at the minimum of the WTE curve, L = tonearm length, d = overhang, M = mounting distance, p = linear offset, R_{01} = inner null radius, R_{02} = outer null radius. All these have units of length.
β = offset angle, TA = tracking angle, TE = tracking error. These have units of angle.

Ensure units are consistent, i.e., the same unit of length is used, e.g., mm, and the same unit of angle is used, e.g., degrees.
Note that **Excel's angular mode is radians**, so conversion to/from degrees is necessary when calculating in degrees in Excel.
To convert: radians = degrees * PI / 180, and degrees = radians * 180 / PI

'LÖFGREN A' EQUATIONS

A. OPTIMUM OFFSET ANGLE

A1. $$\sin\beta = \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1 R_2\right]}$$ Löfgren

A2. $$= \frac{Ra^2}{L\,Rw}$$ Löfgren

A3. $$= \frac{Ro1 + Ro2}{2L}$$ Stevenson

B. OPTIMUM OVERHANG

B1. $$d = L - \left[L^2 - \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1 R_2}\right]^{\frac{1}{2}}$$ Löfgren

B2. $$= L - (L^2 - Ra^2)^{\frac{1}{2}}$$ Löfgren

B3. $$= L - (L^2 - Ro1\,Ro2)^{\frac{1}{2}}$$ Löfgren

B4. $$= L - M$$ Wilson

B5. $$= (M^2 + Ra^2)^{\frac{1}{2}} - M$$ Löfgren

B6. and $$L = (M^2 + Ra^2)^{\frac{1}{2}}$$ Löfgren

C. ZERO TRACKING ERROR OR NULL RADII

C1. $$R_{01} = \frac{2R_1R_2}{\left(1-\frac{1}{\sqrt{2}}\right)R_1 + \left(1+\frac{1}{\sqrt{2}}\right)R_2}$$ Baerwald

C2. $$R_{02} = \frac{2R_1R_2}{\left(1+\frac{1}{\sqrt{2}}\right)R_1 + \left(1-\frac{1}{\sqrt{2}}\right)R_2}$$ Baerwald

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

D1. $$R_W = \frac{2R_1R_2}{R_1+R_2}$$ Löfgren

D2. $$R_W = \frac{L^2-M^2}{L\sin\beta}$$ Löfgren

D3. $$R_W = \frac{2R_{01}R_{02}}{R_{01}+R_{02}}$$ Stevenson

D4. $$R_W = \frac{2Ld-d^2}{L\sin\beta}$$ Löfgren

D5. $$R_W = \frac{Ra^2}{L\sin\beta}$$ Löfgren

E. RADIUS OF MAXIMUM TRACKING ERROR BETWEEN NULL RADII (Ra)

E1. $Ra^2 = L^2 - M^2$ Löfgren

E2. $$= \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2}$$ Löfgren

E3. $= 2Ld - d^2$ Löfgren

E4 $= Ro1 . Ro2$ Löfgren

F. TRACKING ANGLE AT RADIUS R

* F1. $$\sin\alpha = \frac{R^2 + L^2 - M^2}{2LR}$$ Harsanyi

F2. $$= \frac{R^2 + 2Ld - d^2}{2LR}$$ Harsanyi

G. TRACKING ERROR

G1. $\phi = \alpha - \beta$ Harsanyi

H. APPROXIMATE WTE AT RADIUS R

H1. $$\frac{\phi}{R} \approx \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right] \cdot \frac{1}{R\cos\beta}$$ Löfgren

in radians per unit length.

Multiply by $\frac{180}{\pi}$ for degrees per unit length.

EQN H1 is equivalent to Löfgren's EQN (33).
Note that $L^2 - M^2 = 2Ld - d^2$

I. APPROXIMATE WTE AT RADIUS R

I1. $$\frac{\phi}{R} \approx \frac{1}{(L^2 - p^2)^{\frac{1}{2}}}\left[\frac{1}{2} - \frac{p}{R} + \frac{Ra^2}{2R^2}\right]$$ Löfgren

in radians per unit length.

Multiply by $\frac{180}{\pi}$ for degrees per unit length.

EQN I1 is equivalent to the WTE part of Löfgren's EQN (45).

NOTES:

1. EQN H1 and EQN I1 are mathematically identical.

2. These equations are accurate to within around 0.1% of the exact WTE given by EQN J on page S9-6.

J. EXACT WTE AT RADIUS R

J1. $$\frac{\phi}{R} = \frac{\mathrm{ASIN}\left(\frac{R^2 + L^2 - M^2}{2LR}\right) - \beta}{R}$$ Löfgren

J2. $$\frac{\phi}{R} = \frac{\mathrm{ASIN}\left(\frac{R^2 + 2Ld - d^2}{2LR}\right) - \beta}{R}$$ Löfgren

K. SHOW THAT $2Ld - d^2 = L^2 - M^2$

$$\begin{aligned} 2Ld - d^2 &= 2L(L-M) - (L-M)^2 \\ &= 2L^2 - 2LM - L^2 + 2LM - M^2 \\ &= L^2 - M^2 \end{aligned}$$

L. MAXIMUM WEIGHTED TRACKING ERROR

L1. $$|WTE|\,\mathrm{max.} = \frac{\frac{\rho}{R_W} - 1}{2\left[L^2 - \rho^2\right]^{\frac{1}{2}}}$$ Löfgren

radians per unit length.

Multiply by $\frac{180}{\pi}$ for degrees per unit length.

M. MOUNTING DISTANCE

M1. $$M = L - d$$

M2. $$= \left[L^2 - Ra^2\right]^{\frac{1}{2}}$$ Löfgren

M3. $$= \left[L^2 - Ro_1 . Ro_2\right]^{\frac{1}{2}}$$ Löfgren

M4. $$= \left[\frac{R_2\left(L^2 + R_1^2\right) - R_1\left(L^2 + R_2^2\right)}{R_2 - R_1}\right]^{\frac{1}{2}}$$

Kessler and Pisha

N. ZERO TRACKING ERROR OR NULL RADII

N1. $$R = p \pm \left[p^2 - (2Ld - d^2)\right]^{\frac{1}{2}}$$ Dennes

N2. $$R = p \pm \left[p^2 - (L^2 - M^2)\right]^{\frac{1}{2}}$$ Dennes

N3. $$R = p \pm \left[p^2 - Ra^2\right]^{\frac{1}{2}}$$ Dennes

N4. where p = Linear offset.

β, d do not have to be optimum

O. LINEAR OFFSET

O1. $$p = L \sin\beta$$ Löfgren

O2. $$= \frac{Ra^2}{Rw}$$ Löfgren

O3. $$= \frac{Ro1 + Ro2}{2}$$ Stevenson

O4. $$= \frac{R_1 R_2 (R_1 + R_2)}{\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2}$$ Löfgren

P. GROOVE RADII IN TERMS OF NULL RADII

P1. $$R1 = \frac{2\,Ro1\,Ro2}{Ro1(1 - \sqrt{2}) + Ro2(1 + \sqrt{2})}$$ Kearns

P2. $$R2 = \frac{2\,Ro1\,Ro2}{Ro1(1 + \sqrt{2}) + Ro2(1 - \sqrt{2})}$$ Kearns

Q. ARM LENGTH AND OVERHANG WHEN M IS FIXED

1. Calculate Ra^2 using EQN E2 on page S9-4.
2. From EQN E1 on page S9-4:

$$L^2 = M^2 + Ra^2$$

so $L = (M^2 + Ra^2)^{\frac{1}{2}}$

and $d = L - M$

R. DERIVATIVE OF THE WTE EXPRESSION

The WTE expression is shown at EQN J1 on page S9-6. The derivative of this expression with respect to the radius R is:

R1. $$\frac{R^2 - L^2 + M^2}{2LR^3(1-u^2)^{\frac{1}{2}}} - \frac{\arcsin(u) - \beta}{R^2}$$

R2. $$\text{where } u = \frac{R^2 + L^2 - M^2}{2LR}$$

for β in radians

NOTE: $R^2 - L^2 + M^2 = R^2 - 2Ld + d^2$

$R^2 + L^2 - M^2 = R^2 + 2Ld - d^2$

THE 'PERFECT LÖFGREN A' SOLUTION

The derivative of the WTE expression is shown at EQN R on page S9-9. This may be used to find the 'perfect Löfgren A' solution (the 'perfect' three-point, equal WTE solution). For this, the radius R_W is calculated per the procedure on page S12-15, while at the same time, the offset angle and overhang values are adjusted by the iterative procedure until the WTE at R_1, R_W, and R_2 are not just very close (as provided by the 'Löfgren A' alignment equations), but are *exactly* equal to the chosen degree of accuracy, allowing for a digit or two of difference.

This method produces the 'perfect Löfgren A' solution, providing the *lowest possible* WTE for the given L, R_1, and R_2 values.

As an example of a solution obtained in this manner, and to indicate the degree of accuracy achievable by this technique, for an arm length L = 250 mm, inner radius R_1 = 60.325 mm, and outer radius R_2 = 146.05 mm, the results below have been obtained for the 'Perfect Löfgren A' alignment. These results are correct to the number of digits presented, and all results have been generated using Microsoft Excel and its Solver tool. (The digits have been grouped to allow for ease of comparison).

The complete solution results are given on page S1-22.

Please note: The purpose of this exercise is not to present *practical* alignment values, but to provide insight into the possibilities for this high accuracy technique, which does *not* make use of the mathematical approximations used by the six authors.

Optimised Alignment Parameters

Offset angle	= 21.962 997 305 1267 degrees
Overhang	= 16.515 729 704 9773 mm
Radius Rw	= 85.388 512 135 6418 mm
Null radius R_{01}	= 65.978 931 035 1002 mm
Null radius R_{02}	= 121.02 493 022 4359 mm

WTE at the Three Peaks

WTE at R_1	= 0.0002 038 948 297 496 17 TAN(TE) per mm
WTE at Rw	= -0.0002 038 948 297 496 18 TAN(TE) per mm
WTE at R_2	= 0.0002 038 948 297 496 16 TAN(TE) per mm

DETERMINE THE "BEST" OVERHANG FOR NON-OPTIMUM OFFSET ANGLE FOR 'LÖFGREN A' ALIGNMENT

For some situations, it is not possible to set the offset angle to the optimum value. This may be because the offset angle is fixed, or because of insufficient adjustment range. In these situations, there is still a "best" overhang value which results in the lowest WTE, although, of course, the WTE will be greater than that obtained when optimum offset angle and overhang are employed.

The following procedure is based on Seagrave [9], Parts 2 and 3, and Bauer [14].

The Procedure

1. Obtain measurements of the arm length L, and the (non-optimum) offset angle β.

2. Calculate the optimum offset angle β_0 using EQN A1 on page S9-2.

3. Calculate the special offset angle β_1 for which the radius of minimum WTE occurs at R1, using the following equation:

$$\sin\beta_1 = \frac{R_1}{L\left[1-\frac{1}{2}\left(1-\frac{R_1}{R_2}\right)^2\right]} \qquad (1)$$

4. If $\beta \geqslant \beta_0$, then

$$D = \frac{R_1}{2}\left[\left[\left(\sin\beta - \frac{R_1}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} + \sin\beta - \frac{R_1}{L}\right] \qquad (2)$$

else if $\beta_1 < \beta < \beta_0$, then

$$D = \frac{R_2}{2}\left[\left[\left(\sin\beta - \frac{R_2}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} + \sin\beta - \frac{R_2}{L}\right] \quad (3)$$

else if $\beta \le \beta_1$, then

$$D = \frac{\sin\beta\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{1}{L}}{\frac{1}{R_1^2} + \frac{1}{R_2^2}} \quad (4)$$

5. The "best" overhang $d = L - (L^2 - 2LD)^{\frac{1}{2}}$ (5)

This overhang will result in the smallest WTE for the given arm length and offset angle.

Background to this Solution

This "best" overhang solution uses the equations from Bauer [14], then applies a correction to maximise the accuracy, as recommended by Seagrave, in that the overhang D as calculated at step 4 above is applied as D1 in EQN (8) on page S5-4 of this analysis. However, as EQN (8) is also an approximation, the accuracy can be further improved by applying D as D1 in EQN (6) on page S5-3, and calculating the overhang d, as in step 5 above.

The derivation of EQNS (2) to (4) is presented in Section 12.

DETERMINE THE "BEST" OFFSET ANGLE FOR NON-OPTIMUM OVERHANG FOR 'LÖFGREN A' ALIGNMENT

Sometimes it is not possible to set the overhang to the optimum value, perhaps because the overhang is fixed, or has insufficient adjustment range. At these times, there is still a "best" offset angle which results in the lowest WTE, although the WTE will be greater than that obtained when the optimum offset angle and overhang are applied.

The following procedure is based on Seagrave [9], Parts 2 and 3, and Bauer [14].

The Procedure

1. Obtain measurements of the arm length L, and the (non-optimum) overhang D. If mounting distance M is given instead of L, then L = M + D.

2. Calculate the optimum overhang, D_0, using EQN B1 on page S9-2.

3. Calculate the special overhang D_1 using the following equation:

$$D_1 = \frac{R_1}{L\left[2\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - R_1\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right]} \qquad (1)$$

4. If $D \geq D_0$, then

$$\sin\beta = 2\left(\frac{2D^2}{R_1^2} + \frac{D}{L}\right)^{\frac{1}{2}} - \frac{2D}{R_1} \qquad (2)$$

else if $D_1 < D < D_0$, then

$$\sin\beta = 2\left(\frac{2D^2}{R_2^2} + \frac{D}{L}\right)^{\frac{1}{2}} - \frac{2D}{R_2} \qquad (3)$$

else if $D \leq D_1$, then

$$\sin\beta = \frac{D\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{1}{L}}{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \qquad (4)$$

5. The "best" offset angle β = arcsin β.

This offset angle will result in the smallest WTE for the given arm length and overhang.

Background to the Solution

This "best" offset angle solution uses the equations from Bauer [14], then applies a correction to maximise the accuracy, as recommended by Seagrave, in that the offset angle β (radians) calculated by Bauer's equations are changed to $\sin\beta$, per the equations in step 4 above.
See Section 12 for the derivation of EQNS (2) to (4).

CALCULATION OF THE MEAN AND RMS DISTORTION

At EQN(46) of his paper, Löfgren provides an equation to calculate the Mean (of the squares) of the distortion created by tracking error, which is primarily second harmonic in nature.

Further, the RMS distortion value may be calculated from the Mean value.

Löfgren's EQN(46) is as follows:

$$\text{MEAN} = \frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - p^2} \cdot \frac{1}{R_2 - R_1} \left[\frac{R_2 - R_1}{4} - p \ln \frac{R_2}{R_1} + \left(\frac{Ra^2}{2} + p^2 \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{p\,Ra^2}{2} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right) + \frac{(Ra^2)^2}{12} \left(\frac{1}{R_1^3} - \frac{1}{R_2^3} \right) \right]$$

where V = maximum peak recorded velocity
Ω = record speed, radians per second

This equation is applicable to any alignment arrangement.

We will do a calculation on the next page.

MEAN AND RMS DISTORTION EXAMPLE

For V = 100 mm per second (peak)
Ω = 3.49065850398866 rad/sec ($33\frac{1}{3}$ RPM)
L = 250 mm
Ra^2 = 7744 mm^2
p = 93.6516483539780
β = 0.383972435438752 radians ($22°$)
d = 16 mm
R_1 = 60.325 mm
R_2 = 146.05 mm

Mean = 0.0000268995210202244

$$RMS = (Mean)^{\frac{1}{2}}$$
$$= (0.0000268995210202244)^{\frac{1}{2}}$$
$$= 0.00518647481631063$$
$$= 0.5186\%$$

Your figures may vary slightly.

NOTE: RIAA CORRECTION NOT INCLUDED.

Note. Not all of the above parameters are used directly in the calculations. They are shown for completeness of the solution.

This is an example of the distortion level calculated by the formula on the previous page, provided by Löfgren's EQN (46).

THE EFFECT OF THE PHONO PREAMP RIAA STAGE ON DISTORTION

The RIAA equalisation stage in the phono preamplifier has an overall effect of rolling off (decreasing) the response as the frequency increases. The frequency roll-off has to be allowed for in distortion calculations.

Löfgren did not allow for the RIAA response (or any other response) in his calculations

For a fundamental frequency of 1 KHz, the standard (three time-constants) RIAA playback curve shows a response of 0.0889815809580368 dB.

For a 2 KHz frequency, it shows a response of -2.49955932706491 dB. Thus the second harmonic at 2 KHz is 2.58854090802295 dB below the fundamental.

This equates to 10^(-2.58854090802295 / 20) or 0.742288880049004 times, meaning the second harmonic at 2 KHz is reduced by that figure, ie, the distortion is lowered by this amount due to the RIAA factor.

The RIAA factor can be applied in either of three ways:

1. As a multiplier for the distortion factor K before K is squared, i.e.
   K = 0.5 * WTE * 0.742288880049004
2. As a squared multiplier for the mean distortion, i.e.
   mean = mean * 0.742288880049004 ^ 2
3. As a multiplier for the RMS distortion, i.e.
   RMS = RMS * 0.742288880049004

The third method will be used here.

Following on from the example on the previous page, where:

RMS = 0.00518647481631063

Then RMS = RMS * 0.742288880049004

= 0.00518647481631063 * 0.742288880049004

= 0.00384986258280157

= 0.384986258280157%

"LÖFGREN B" EQUATIONS

A. OFFSET ANGLE

The offset angle is an input to the "Löfgren B" alignment, not an output.

B. OVERHANG

B1. $$d = L - \left[L^2 - Ra^2\right]^{\frac{1}{2}}$$ Löfgren

B2. $$= L - \left[L^2 - R_{01}R_{02}\right]^{\frac{1}{2}}$$ Löfgren

B3. $$= L - M$$

B4. where $$Ra^2 = \frac{3R_1R_2\left[p(R_1+R_2) - R_1R_2\right]}{(R_1+R_2)^2 - R_1R_2}$$ Löfgren

B5. and $$p = L \sin \beta$$ Löfgren

C. ZERO TRACKING ERROR OR NULL RADII

C1. $R = \rho \pm \left[\rho^2 - (2Ld - d^2) \right]^{\frac{1}{2}}$ Dennes

C2. $\quad = \rho \pm \left[\rho^2 - (L^2 - M^2) \right]^{\frac{1}{2}}$ Dennes

C3. $\quad = \rho \pm \left[\rho^2 - Ra^2 \right]^{\frac{1}{2}}$ Dennes

where ρ = linear offset

β, d do not have to be optimum.

D. RADIUS OF MAXIMUM WTE BETWEEN NULL RADII

D1. $Rw = \dfrac{L^2 - M^2}{L \sin\beta}$ Löfgren

D2. $\quad = \dfrac{2Ld - d^2}{L \sin\beta}$ Löfgren

D3. $\quad = \dfrac{Ra^2}{L \sin\beta}$ Löfgren

D4. $\quad = \dfrac{2\, Ro_1 Ro_2}{Ro_1 + Ro_2}$ Stevenson

β, d do not have to be optimum.

E. RADIUS OF MAXIMUM TRACKING ERROR BETWEEN NULL RADII (Ra)

E1. $$Ra^2 = L^2 - M^2$$ Lötgren

E2. $$= 2Ld - d^2$$ Löfgren

E3. $$= R_{01} . R_{02}$$ Lötgren

E4. $$= \frac{3 R_1 R_2 \left[P (R_1 + R_2) - R_1 R_2 \right]}{(R_1 + R_2)^2 - R_1 R_2}$$ Lötgren

E5. where p = Linear offset

E6. $$= L \sin \beta$$ Lötgren

where β is specified offset angle

F. MOUNTING DISTANCE

F1. $$M = \left[L^2 - Ra^2\right]^{\frac{1}{2}}$$ Löfgren

$$= \left[L^2 - Ro_1 Ro_2\right]^{\frac{1}{2}}$$ Löfgren

G. LINEAR OFFSET

G1. $$p = L \sin\beta$$ Löfgren

G2. $$= \frac{Ro_1 + Ro_2}{2}$$ Stevenson

G3. $$= \frac{Ra^2}{Rw}$$ Löfgren

H. ARM LENGTH AND OVERHANG WHEN M IS FIXED

1. Calculate Ra^2 using EQNs E4 and E7 on page S9-20.

2. From EQN E1 on page S9-20:

$$L^2 = M^2 + Ra^2$$

so $$L = (M^2 + Ra^2)^{\frac{1}{2}}$$

and $$d = L - M$$

'LÖFGREN C' ALIGNMENT EQUATIONS

Background

For the 'Löfgren C' alignment, the offset angle and overhang are adjusted for minimum LMS/RMS distortion between radii R_1 and R_2. Although Löfgren partially solved this alignment by providing the solution for the optimum overhang d (as also used in the 'Löfgren B' alignment, introducing the auxiliary parameter Ra^2 in the process), Löfgren did not solve for the optimum offset angle β.

New, Historic Formula

A new, historic formula-based solution for the 'Löfgren C' alignment has been made possible by the provision of a formula for the linear offset *p* by Vladan Jovanovic [18], shown below and at EQN (27) in [18]. The formula was derived by finding partial derivatives of the distortion integral with respect to offset angle β and overhang d, equating them to zero and solving the two resulting equations for the two unknowns β and d.

This new solution for *p* enables the optimum offset angle β, shown below and at EQN (29) in [18], to be calculated, finally enabling an accurate formula-based solution to be applied to the 'Löfgren C' alignment. This is a major step forward in tonearm alignment geometry and tonearm alignment history. In fact, it is the first new *optimum* alignment formula in the 83 years since Löfgren's ground-breaking article was published! This is the formula Löfgren did not provide.

The value of *p* is then substituted into Löfgren's EQN (48) to calculate Ra^2, shown below and at EQN (28) in [18], after which the value of Ra^2 is substituted into Löfgren's EQN (47), shown below and at EQN (30) in [18], to calculate the optimum overhang d.

The zero error radii R_{01} and R_{02} are calculated by substituting the values for *p* and Ra^2 into the standard quadratic solution, shown below and at EQN (7) on page S12-11 and EQN (31) in [18].

The resulting alignment figures provide excellent agreement with the Excel-based 'perfect Löfgren C' alignment figures shown elsewhere in this book.

Jovanovic's Formula for *p*

$$p = \frac{(L^2 + \mathrm{R}_1\mathrm{R}_2)(\mathrm{R}_2 - \mathrm{R}_1)^3 - \sqrt{(\mathrm{R}_2 - \mathrm{R}_1)^6(L^2 + \mathrm{R}_1\mathrm{R}_2)^2 - 4L^2\mathrm{R}_1^2\mathrm{R}_2^2\left[2(\mathrm{R}_1^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2^2)\log\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right) - 3(\mathrm{R}_2^2 - \mathrm{R}_1^2)\right]^2}}{4\mathrm{R}_1\mathrm{R}_2(\mathrm{R}_1^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2^2)\log\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right) - 6\mathrm{R}_1\mathrm{R}_2(\mathrm{R}_2^2 - \mathrm{R}_1^2)}$$

where L = tonearm length and R_1 and R_2 = inner and outer recorded groove radii respectively. When using Excel to perform the calculation, use Excel's *LN* (natural logarithm) function.

Optimum Offset Angle β

β = ASIN(*p* / L). For Excel, multiply β by 180 / PI() to convert from radians to degrees.

Löfgren's EQN (48) for Ra^2

$$Ra^2 = \frac{3R_1R_2[\,p\,(R_1 + R_2)\; - \;R_1R_2\,]}{(R_1 +\; R_2)^2 - R_1R_2}$$

Löfgren's EQN (47) for Overhang d

d = L – SQRT (L^2 – Ra^2)

Zero Error Radii

R_{01}, R_{02} = *p* ∓ SQRT (p^2 – Ra^2)

ANALYTICAL SOLUTION TO 'LÖFGREN C' ALIGNMENT EQUATIONS

Background
In a recent paper by Peet Hickman [19], the offset angle and overhang solutions for the 'Löfgren A', 'Löfgren B' and 'Löfgren C' alignments are fully derived from first principles. This is an historic first. This includes the linear offset *p* formula for the 'Löfgren C' alignment, from which the offset angle is calculated. (Jovanovic [18] was the first to present the linear offset formula for the 'Löfgren C' alignment.)

Hickman's 'Löfgren C' linear offset *p* formula is presented in [19] at EQN (56), with an auxiliary parameter p_∞ formula presented at EQN (45). These formulas follow. Hickman also proves the equivalence of Jovanovic's linear offset formula with his own.

Hickman's Formula for Linear Offset *p*

EQN (56):

$$p = \frac{2p_\infty}{\left(1 + \frac{R_1 R_2}{L^2}\right) + \sqrt{\left(1 + \frac{R_1 R_2}{L^2}\right)^2 - \frac{4p_\infty^2}{L^2}}}$$

with EQN (45):

$$p_\infty = \frac{R_1 R_2 \, [2(R_1^2 + R_1 R_2 + R_2^2)\ \mathrm{LN}(R_2/R_1) - 3(R_2^2 - R_1^2)]}{(R_2 - R_1)^3}$$

where L = effective tonearm length, R_1 and R_2 = inner and outer recorded groove radii respectively, and LN is the natural logarithm function.

The Remaining 'Löfgren C' Alignment Equations
The remaining 'Löfgren C' alignment equations are as per page S9-22.

'STEVENSON A' EQUATIONS

A. OFFSET ANGLE

Inputs: L, R_{01}, R_2

A1. $$\sin\beta = \frac{R_a^2}{L R_W}$$ Löfgren

A2. where $$R_a^2 = \frac{R_{01}^2 R_2}{(2\sqrt{2}-2)R_{01} - (2\sqrt{2}-3)R_2}$$ Stevenson

A3. and $$R_W = \frac{2 R_{01} R_a^2}{R_{01}^2 + R_a^2}$$ Stevenson

B. OVERHANG

B1. $$d = L - \left(L^2 - R_a^2\right)^{\frac{1}{2}}$$ Löfgren

where R_a^2 = as above

C. OUTER NULL RADIUS

C1. $$R_{02} = \frac{R_{01} R_2}{(2\sqrt{2}-2)R_{01} - (2\sqrt{2}-3)R_2}$$ Stevenson

C2. $$= \frac{R_a^2}{R_{01}}$$ where R_a^2 = as above. Löfgren

D. LINEAR OFFSET

D1. $$p = L\sin\beta$$ Löfgren

DETAILED DERIVATION OF LÖFGREN'S 'LÖFGREN A' ALIGNMENT EQUATIONS

This Section presents the detailed derivation of the alignment equations presented by Löfgren for the 'Löfgren A' alignment.

1.	EQN (2)	Tracking angle
2.	EQN (22)	Tracking distortion
3.	EQN (32)	Tracking error
4.	EQN (33)	Weighted tracking error
5.	EQN (34)	Optimum offset angle
6.	EQN (36) and (37)	Minimum weighted tracking error radius R_W
7.	EQN (39)	Optimum overhang
8.	EQN (40) and (39)	Minimum angular tracking error radius R_a
9.	EQN (41) and (42)	Linear offset
10.	EQN (43)	Maximum WTE
11.	EQN (35)	WTE at R_1 and R_2
12.	EQN (38)	WTE at R_W

The equation numbering used in the derivation of Löfgren's EQN (22) are the same numbers used in Löfgren's paper. The equation numbering used in the derivation of the remaining equations above are in sequential order for ease of reference.

NOTATION USED

β	Offset angle
d	Overhang
L	Effective arm length
M	Mounting distance
	= L - d
R_1	Inner recorded groove radius
R_2	Outer recorded groove radius
R_{01}	Inner null radius
R_{02}	Outer null radius
R_a	Radius of greatest (angular) lateral tracking error between the null radii
R_w	Radius of greatest weighted tracking error between the null radii
R	any radius
Ω	Turntable angular velocity
P	Linear offset
V	Recorded velocity
ε	Distortion
α	Lateral tracking angle
φ	Lateral tracking error
	$= \alpha - \beta$
γ	angle complementary to α
γ_0	γ when $\varphi = 0$
	γ_0, β are complementary
O	Turntable centre
P	Tonearm pivot
S	Stylus position

BASIC TONEARM GEOMETRY

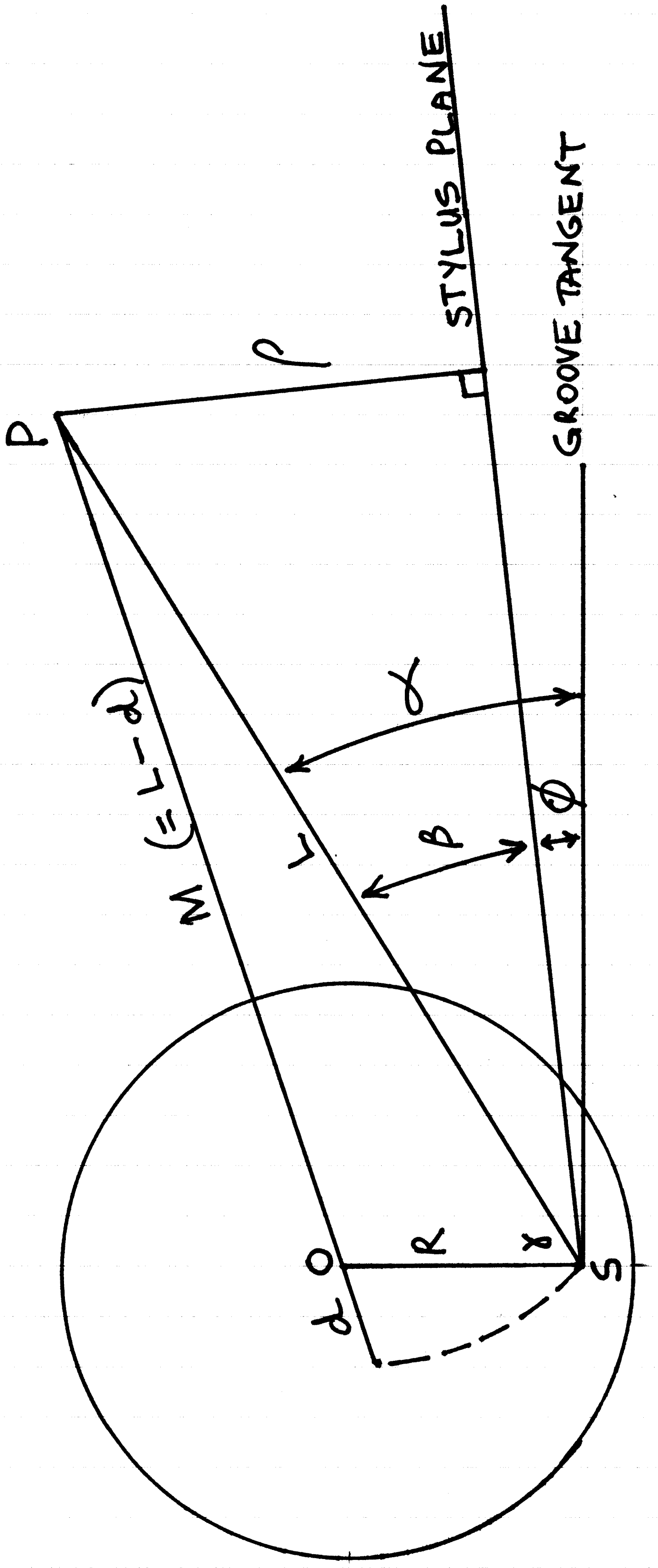


FIG. 1

GRAEME F. DENNES

MATHEMATICAL APPROXIMATIONS USED BY LÖFGREN

Löfgren used the following mathematical approximations in his paper to enable the 'Löfgren A' and 'Löfgren B' solutions to be obtained.

1. In the derivation of his EQN (22):

   For δ small, and δ in radians, then $\tan\delta \approx \delta$.

2. In the derivation of his EQN (32):

   For ϕ small, and ϕ in radians, then
   $\cos\phi \approx 1$ and
   $\sin\phi \approx \phi$

The Later Authors

All the later authors used the same or similar approximations for the same purpose Löfgren used them: to enable their alignment equations to be obtained.

DERIVATION OF THE TRACKING ANGLE EQUATION
LÖFGREN'S EQN (2)

Consider FIG. 1.

In the triangle OSP, from the Cosine Rule:

$$(L-d)^2 = L^2 + R^2 - 2LR\cos\gamma \qquad (1)$$

As γ and α are complementary, then

$$(L-d)^2 = L^2 + R^2 - 2LR\sin\alpha \qquad (2)$$

So $$\sin\alpha = \frac{R^2 + L^2 - (L-d)^2}{2LR} \qquad (3)$$

$$= \frac{R^2 + 2Ld - d^2}{2LR} \qquad (4)$$

As $d = L - M$

then $$\sin\alpha = \frac{R^2 + 2L(L-M) - (L-M)^2}{2LR} \qquad (5)$$

$$= \frac{R^2 + L^2 - M^2}{2LR} \qquad (6)$$

EQNS (4) and (6) are equivalent.

EQN (6) is identical to Löfgren's EQN (2).

DERIVATION OF LÖFGREN'S DISTORTION EQUATION (LÖFGREN'S EQN (22))

An expression for the level of second harmonic distortion is presented at EQN (22) of Löfgren's paper, and shows the relationship between the parameters involved.

$$\varepsilon \approx \frac{V}{\Omega} \cdot \frac{\delta}{R} \quad * \tag{22}$$

where ε = second harmonic level
V = peak recorded groove velocity
Ω = turntable angular velocity
δ = tracking error
R = groove radius

This equation formed the basis of the research undertaken by Löfgren, and its derivation follows. The equations and their reference numbers which follow match those used in Löfgren's paper.

Löfgren was the first to unify the variables in the manner shown in EQN (22).

* Ensure the units are consistent.
If V = mm per sec, then R = mm, etc.
If δ = radians, then Ω = radians per sec, etc.

The summarised derivation of Löfgren's EQN (22) which follows is provided here for background and understanding.

From Löfgren's Fig. 3, the stylus deflection z in the presence of tracking error δ is:

(7) $$z = \frac{A}{\cos\delta} \cdot \sin \frac{2\pi}{\lambda}\left(x - z\sin\delta\right)$$

Letting $\theta = \frac{2\pi}{\lambda}\left(x - z\sin\delta\right)$, then

(12) $$z = \frac{A}{\cos\delta} \cdot \sin\theta$$

Now, $\theta = \frac{2\pi}{\lambda}\left(x - z\sin\delta\right)$ from above

$$= \frac{2\pi x}{\lambda} - \frac{2\pi z}{\lambda}\sin\delta$$

$$= \frac{2\pi x}{\lambda} - \frac{2\pi}{\lambda} \cdot \frac{A}{\cos\delta} \cdot \sin\delta \cdot \sin\theta$$

$$= \frac{2\pi x}{\lambda} - \frac{2\pi A}{\lambda} \cdot \tan\delta \cdot \sin\theta$$

or $$\frac{2\pi x}{\lambda} = \theta + \frac{2\pi A}{\lambda} \cdot \tan\delta \cdot \sin\theta$$

or

(13) $$\frac{2\pi x}{\lambda} = \theta + \varepsilon \sin\theta$$

where

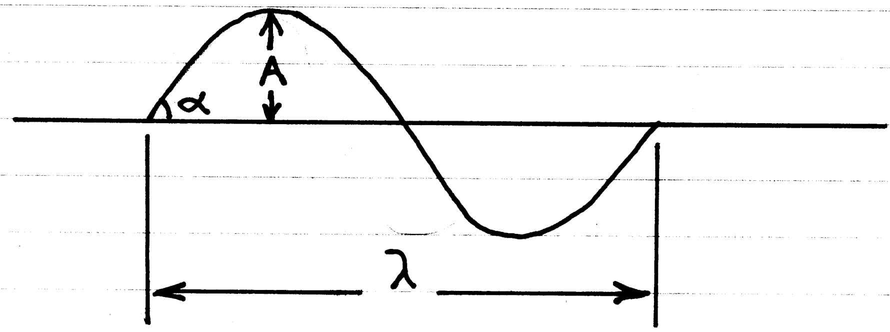
(14) $$\varepsilon = \frac{2\pi A}{\lambda} \cdot \tan\delta$$

where A = physical groove amplitude
λ = physical groove wavelength
δ = tracking error

The term ε represents distortion, and can also be expressed as:

(14a) $\varepsilon = \tan\alpha \tan\delta$

where α is the angle of maximum physical gradient of the groove modulation:



Olney [5] postulated that the distortion from tracking error would be related to the ratio of the amplitude of the groove modulation to the wavelength of the groove modulation, ie, $\frac{A}{\lambda}$. Löfgren confirmed this at EQN (14).

Löfgren uses a Fourier series for the stylus deflection z in the presence of tracking error, and develops a Bessel Function power series to calculate the relative velocities of the fundamental and the harmonics.

(19a)
$$V_n = \frac{V}{\cos\delta} \cdot \frac{(n\,\varepsilon)^{n-1}}{2^{n-1}(n-1)!}\left\{1 - \frac{\left(\frac{n\,\varepsilon}{2}\right)^2}{1!\,(n+1)} + \frac{\left(\frac{n\,\varepsilon}{2}\right)^4}{2!\,(n+1)(n+2)} - \dots\right\}.$$

where V_1 = fundamental velocity
V_2 = second harmonic velocity
V_3 = third harmonic velocity
V = stylus velocity with tangential tracking (zero tracking error)

For the fundamental and the fist two harmonics, we obtain the velocities:

$$V_1 = \frac{V}{\cos\delta}\cdot\left(1-\frac{\varepsilon^2}{8}+\frac{\varepsilon^4}{192}-\cdots\right),$$

$$\text{(19b)}\quad V_2 = \frac{V}{\cos\delta}\cdot\varepsilon\left(1-\frac{\varepsilon^2}{3}+\frac{\varepsilon^4}{24}-\cdots\right),$$

$$V_3 = \frac{V}{\cos\delta}\cdot\frac{9\,\varepsilon^2}{8}\left(1-\frac{9\,\varepsilon^2}{16}+\frac{81\,\varepsilon^4}{640}-\cdots\right).$$

Apart from the frequency response influences of the playback equipment, etc, the frequency components in the sound wave are present in the same amplitude ratios as they are in the groove velocities above, ie, the amplitude ratios are the same as the velocity ratios.

Now we'll estimate the maximum gradient angle α of the groove modulation. With Ω being the angular velocity of the record, then the wavelength λ at angular frequency ω and radius R is:

$$\text{(20)}\quad \lambda = \frac{2\pi\,\Omega R}{\omega}$$

And so from (14) and (14a),

$$\tan\alpha = \frac{2\pi A}{\lambda} = 2\pi A.\frac{\omega}{2\pi\Omega R} = \frac{A\omega}{\Omega R} = \frac{V}{\Omega R} \quad (21)$$

The distortion factor at (14a) can now be shown by:

$$\varepsilon = \frac{V}{\Omega R}\tan\delta$$

Further, on condition that the tracking error δ is small, then $\tan \delta$ can be replaced by δ (radians) to a good approximation, which results in an expression for ε in a form better suited to our needs:

$$(22) \qquad \varepsilon \approx \frac{V}{\Omega} \cdot \frac{\delta}{R}$$

where ε = distortion factor
V = recorded groove velocity
Ω = turntable angular velocity
δ = tracking error
R = groove radius

As Löfgren notes, the first factor is independent of the position of the needle on the record, while the second factor changes continuously.

Consider an example. Ensure the units are consistent, as noted previously.

Let V = 100 mm/sec
$\Omega = 2\pi \times 78/60$ = 8.2 radians per sec
$\delta = 11° = 0.192$ radians
R = 50 mm

Using EQN (19b), calculate the velocities of the fundamental (V_1), the second harmonic (V_2) and the third harmonic (V_3).

From EQN (22), $\varepsilon = 0.047$

Then, V_1 = 101.8 mm/sec (fundamental)
V_2 = 4.78 mm/sec (second harmonic)
V_3 = 0.253 mm/sec (third harmonic)

Now, the ratio of the second harmonic (V_2) to the fundamental (V_1) is:

$$\frac{V_2}{V_1} = \frac{4.78}{101.8} = 0.047 = 4.7\%$$

The ratio of the third harmonic (V_3) to the fundamental (V_1) is:

$$\frac{V_3}{V_1} = \frac{0.253}{101.8} = 0.0025 = 0.25\%$$

As the third harmonic (V_3) is only 0.25% of the fundamental (V_1), we may ignore it.

Thus, the main distortion component is primarily second harmonic in nature (V_2).

We note that $\frac{V_2}{V_1} = 0.047$ from above.

We also note that $\varepsilon = 0.047$ from EQN (22).

Conclusions

1. The distortion factor $\frac{V_2}{V_1}$ can be set to ε.

2. The second harmonic V_2 can be set to $V_2 = \varepsilon V_1$ with good approximation.

3. The higher harmonics can be neglected.

4. Löfgren's EQN (22) is an excellent approximation to the distortion factor.

DERIVATION OF TRACKING ERROR EQUATION
LÖFGREN'S EQN (32)

As Löfgren's EQN (22) shows, tracking distortion is proportional to the tracking error, and inversely proportional to the groove radius.

From FIG. 1, the tracking error ϕ is:

$$\phi = \alpha - \beta \qquad (7)$$

where α = tracking angle
β = offset angle

As $$\alpha = \arcsin\left[\frac{R^2 + 2Ld - d^2}{2LR}\right] \qquad (8)$$

from EQN (4),

then to minimise tracking distortion, we need to minimise ϕ, where

$$\phi = \arcsin\left[\frac{R^2 + 2Ld - d^2}{2LR}\right] - \beta \qquad (9)$$

To minimise ϕ, we need to analyse the variables involved. While EQN(9) is exact, it does not lend itself to simple analytic treatment. The problem is the arcsin function, in that the variables R, L and d cannot be isolated or separated for analysis.

Perhaps we can find an approximation to ϕ which will be of sufficient accuracy for our needs, and allow "access" to the variables R, L and d.

Refering again to FIG. 1.

Let $\gamma = \frac{\pi}{2} - \beta - \phi$, and $\gamma_0 = \gamma$ when $\phi = 0$,

so $\gamma = \gamma_0 - \phi$ always (10)

Now, we need an expression for ϕ, as tracking distortion is proportional to ϕ.

From (10), and taking cosines:

$$\cos\gamma = \cos(\gamma_0 - \phi) \qquad (11)$$

From EQN (1), and the Cosine Rule,

$$(L-d)^2 = L^2 + R^2 - 2LR\cos\gamma \qquad (12)$$

$$\text{so } \cos\gamma = \frac{R^2 + 2Ld - d^2}{2LR} \qquad (13)$$

Using the standard substitution:

$$\cos(a-b) = \cos a\cos b + \sin a\sin b \qquad (14)$$

Letting $a = \gamma_0$

and $b = \phi$, we have from (11) and (14):

$$\cos\gamma = \cos(\gamma_0 - \phi)$$

$$\text{and } \cos(\gamma_0 - \phi) = \cos\gamma_0\cos\phi + \sin\gamma_0\sin\phi \qquad (15)$$

This equation is exact in every respect. However, by using approximations, we can achieve the expansion of ϕ, as follows.

For ϕ small, and ϕ and γ_0 in radians:

$\cos\phi \approx 1$ and $\sin\phi \approx \phi$, so from (15):

$$\cos\gamma \approx \cos\gamma_0 . 1 + \sin\gamma_0 . \phi$$

and $$\phi \approx \frac{\cos\gamma - \cos\gamma_0}{\sin\gamma_0} \quad \text{radians}$$

We now have an expression for ϕ on its own.

Noting that γ_0 and β are complementary, then

$$\phi \approx \frac{\cos\gamma - \sin\beta}{\cos\beta} \quad \text{radians} \qquad (16)$$

We also note that $\phi = \alpha - \beta$, so

$$\alpha - \beta \approx \frac{\cos\gamma - \sin\beta}{\cos\beta} \quad \text{radians}$$

From (13), $$\cos\gamma = \frac{R^2 + 2Ld - d^2}{2LR}$$

$$= \frac{R^2 + L^2 - M^2}{2LR}$$

so $$\phi = \alpha - \beta = \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta} \qquad (17)$$

$$= \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta} \qquad (18)$$

radians

EQN (17) and EQN (18) are equivalent, and EQN (18) is the tracking error component of Lötgren's EQN (33).

EQN (16) is equivalent to Lötgren's EQN (32).

DERIVATION OF WEIGHTED TRACKING ERROR EQUATION
LÖFGREN'S EQN (33)

From Löfgren's EQN (22), tracking distortion is proportional to tracking error and inversely proportional to radius, ie tracking distortion is proportional to the weighted tracking error (WTE).

We have:

$$\text{WTE} = \frac{\text{tracking error}}{\text{radius}}$$

$$= \frac{\alpha - \beta}{R} \qquad (19)$$

$$\text{ie WTE} = \frac{\phi}{R} \qquad (20)$$

From EQN (17), we have:

$$\phi \approx \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta}$$

radians

and so

$$\text{WTE} \approx \frac{\left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta}}{R}$$

$$\text{ie WTE} \approx \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (21)$$

radians per unit length

Using EQN (18), we also have:

$$WTE \approx \left[\frac{R^2+L^2-M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (22)$$

radians per unit length

Equations (21) and (22) are equivalent, and EQN (22) is identical to Lötgren's EQN (33).

DERIVATION OF OPTIMUM OFFSET ANGLE LÖFGREN'S EQN (34)

For the 'Löfgren A' alignment, we have the two independent variables of offset angle and overhang which we need to determine. To solve for two independent variables, we need two independent equations in those variables. This section establishes the two equations and solves for the offset angle. The overhang, Löfgren's EQN (39), is solved further on.

From EQN (22):

$$WTE \approx \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (23)$$

radians per unit length

$$\approx \left[\frac{1}{2L} + \frac{L^2 - M^2}{2LR^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta} \qquad (24)$$

For the optimum solution, we require:

1. WTE at R_1 = WTE at R_2

So from (24) we have:

$$\left[\frac{1}{2L} + \frac{L^2 - M^2}{2LR_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} = \left[\frac{1}{2L} + \frac{L^2 - M^2}{2LR_2^2} - \frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} \qquad (25)$$

The 2L and $\cos\beta$ terms drop out, so

$$\frac{L^2 - M^2}{2LR_1^2} - \frac{\sin\beta}{R_1} = \frac{L^2 - M^2}{2LR_2^2} - \frac{\sin\beta}{R_2} \qquad (26)$$

Now solve for $\dfrac{L^2-M^2}{L}$

ie $$\frac{L^2-M^2}{2LR_1^2} - \frac{L^2-M^2}{2LR_2^2} = \frac{\sin\beta}{R_1} - \frac{\sin\beta}{R_2} \qquad (27)$$

ie $$\frac{L^2-M^2}{L}\left[\frac{1}{2R_1^2} - \frac{1}{2R_2^2}\right] = \sin\beta\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

ie $$\frac{L^2-M^2}{L}\left[\frac{2(R_2^2-R_1^2)}{4R_1^2R_2^2}\right] = \sin\beta\left[\frac{R_2-R_1}{R_1R_2}\right]$$

ie $$\frac{L^2-M^2}{L}\left[\frac{2(R_2-R_1)(R_2+R_1)}{4R_1^2R_2^2}\right] = \sin\beta\left[\frac{R_2-R_1}{R_1R_2}\right]$$

so $$\frac{L^2-M^2}{L} = \sin\beta\left[\frac{R_2-R_1}{R_1R_2}\right]\cdot\frac{4R_1^2R_2^2}{2(R_2-R_1)(R_2+R_1)}$$

$$= \sin\beta\cdot\frac{2R_1R_2}{R_1+R_2}$$

ie $$\frac{L^2-M^2}{L} = \frac{2R_1R_2\sin\beta}{R_1+R_2} \qquad (28)$$

We also require:

2. <u>WTE at R1 + WTE at Rw = 0</u>

At the optimum solution, the WTE at R1 and Rw are equal in magnitude but opposite in sign.

Firstly we need to determine the radius R_W, the radius at the minimum of the WTE function. We do this by differentiating the WTE expression with respect to the radius R, equate the derivative to zero, and solve for $R = R_W$.

From (24):

$$WTE \approx \frac{1}{2L\cos\beta} + \frac{L^2-M^2}{2LR^2\cos\beta} - \frac{\sin\beta}{R\cos\beta}$$

so $$\frac{d}{dR} = 0 - \frac{2(L^2-M^2)}{2LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta}$$

ie $$-\frac{(L^2-M^2)}{LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta} = 0 \qquad (29)$$

Multiply by $R^3\cos\beta$:

ie $$-\frac{L^2-M^2}{L} + R\sin\beta = 0 \qquad (30)$$

so $$R\sin\beta = \frac{L^2-M^2}{L}$$

and $$R = \frac{L^2-M^2}{L\sin\beta}$$

ie $$R_W = \frac{L^2-M^2}{L\sin\beta} \qquad (31)$$

Now substitute $R = R_W$ back into (24):

so WTE at R_W =

$$\left[\frac{1}{2L} + \frac{L^2-M^2}{2L}\cdot\frac{(L\sin\beta)^2}{(L^2-M^2)^2} - \sin\beta\cdot\frac{L\sin\beta}{L^2-M^2}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{1}{2L} - \frac{L\sin^2\beta}{2(L^2-M^2)}\right]\frac{1}{\cos\beta} \qquad (32)$$

As we need WTE at R_1 + WTE at R_W = 0, then

$$\left[\frac{1}{2L} + \frac{L^2-M^2}{2LR_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} + \left[\frac{1}{2L} - \frac{L\sin^2\beta}{2(L^2-M^2)}\right]\frac{1}{\cos\beta} = 0 \qquad (33)$$

$$\text{ie } \frac{1}{L} + \frac{L^2-M^2}{2LR_1^2} - \frac{\sin\beta}{R_1} - \frac{L\sin^2\beta}{2(L^2-M^2)} = 0 \qquad (34)$$

Now substitute the expression (28) for $\frac{L^2-M^2}{L}$:

$$\frac{1}{L} + \frac{2R_1R_2\sin\beta}{2R_1^2(R_1+R_2)} - \frac{\sin\beta}{R_1} - \frac{(R_1+R_2)\sin^2\beta}{4R_1R_2\sin\beta} = 0$$

$$\frac{1}{L} + \frac{R_2\sin\beta}{R_1(R_1+R_2)} - \frac{\sin\beta}{R_1} - \frac{(R_1+R_2)\sin\beta}{4R_1R_2} = 0 \qquad (35)$$

Solving for $\sin\beta$:

$$\sin\beta\left[\frac{R_2}{R_1(R_1+R_2)} - \frac{1}{R_1} - \frac{R_1+R_2}{4R_1R_2}\right] = -\frac{1}{L}$$

Expand the brackets to:

$$-\left[\frac{R_1^2 + 6R_1R_2 + R_2^2}{4R_1^2R_2 + 4R_1R_2^2}\right]$$

so $$-\sin\beta\left[\frac{R_1^2 + 6R_1R_2 + R_2^2}{4R_1^2R_2 + 4R_1R_2^2}\right] = -\frac{1}{L}$$

or $$\sin\beta = \frac{4R_1^2R_2 + 4R_1R_2^2}{L\left[R_1^2 + 6R_1R_2 + R_2^2\right]}$$

ie $$\sin\beta = \frac{R_1R_2(R_1+R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \qquad (36)$$

EQN (36) is the standard 'Löfgren A' offset angle equation, and has been derived from first principles.

Continue to next page.

The standard offset angle equation, EQN (36), may be factored into three components, as follows:

From (36):

$$\sin\beta = \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \cdot \frac{1}{2L} \cdot \frac{R_1+R_2}{R_1R_2} \qquad (36A)$$

$$= Ra^2 \cdot \frac{1}{2L} \cdot \left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$= \frac{Ra^2}{2L}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] \qquad (36B)$$

From EQN (65):

$$Ra^2 = L^2 - M^2$$

so $$\sin\beta = \frac{L^2 - M^2}{2L}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] \qquad (36C)$$

The equation above is identical to Löfgren's EQN (34), and has been derived from first principles.

This completes the derivation of the offset angle equation at EQN (36), and Löfgren's EQN (34), for the 'Löfgren A' alignment.

DERIVATION OF RADIUS R_W
LÖFGREN'S EQNS (36) & (37)

The radius R_W is the radius at which the WTE reaches its most negative value, ie, is the radius at the minimum of the WTE function.

For the optimum 'Löfgren A' solution, the WTE at R_1, R_W and R_2 have equal magnitudes.

To determine R_W, we differentiate the WTE equation, equate it to zero, and solve for R, the radius R_W.

From EQN (21),

$$WTE \approx \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta}$$

$$\approx \left[\frac{R^2}{2LR} + \frac{2Ld}{2LR} - \frac{d^2}{2LR} - \frac{\sin\beta}{2LR}\right]\frac{1}{R\cos\beta}$$

$$\approx \frac{1}{2L\cos\beta} + \frac{d}{R^2\cos\beta} - \frac{d^2}{2LR^2\cos\beta} - \frac{\sin\beta}{R\cos\beta} \qquad (37)$$

$$\text{So } \frac{d}{dR} = 0 - \frac{2d}{R^3\cos\beta} + \frac{2d^2}{2LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta} = 0 \qquad (38)$$

Multiply by $R^3\cos\beta$:

$$\text{ie} \quad -2d + \frac{d^2}{L} + R\sin\beta = 0 \qquad (39)$$

$$\text{So} \quad R\sin\beta = 2d - \frac{d^2}{L}$$

$$= \frac{2Ld - d^2}{L}$$

and $R = R_W = \dfrac{2Ld - d^2}{L \sin\beta}$ (40)

Also, $d = L - M$,

so $R_W = \dfrac{2L(L-M) - (L-M)^2}{L \sin\beta}$

$= \dfrac{2L^2 - 2LM - L^2 + 2LM - M^2}{L \sin\beta}$

ie $R_W = \dfrac{L^2 - M^2}{L \sin\beta}$ (41)

(40) and (41) are identical to Löfgren's EQN (36), and (40) and (41) are equivalent.

From EQN (65):

As $L^2 - M^2 = R_a^2$

then $R_W = \dfrac{R_a^2}{L \sin\beta}$ (42)

$= \dfrac{8 R_1^2 R_2^2}{(R_1+R_2)^2 + 4R_1R_2} \cdot \dfrac{1}{L} \cdot \dfrac{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}{R_1R_2(R_1+R_2)}$

$= \dfrac{8 R_1^2 R_2^2}{(R_1+R_2)^2 + 4R_1R_2} \cdot \dfrac{(R_1+R_2)^2 + 4R_1R_2}{4R_1R_2(R_1+R_2)}$

$= \dfrac{2R_1R_2}{R_1+R_2}$ (43)

EQN (43) is identical to Löfgren's EQN (37), and applies when β, d are optimum.

DERIVATION OF OPTIMUM OVERHANG
LÖFGREN'S EQN (39)

From (28):

$$\frac{L^2 - M^2}{L} = \frac{2R_1R_2 \sin\beta}{R_1 + R_2} \tag{44}$$

Now, $M = L - d$, so

$$\frac{L^2 - (L-d)^2}{L} = \frac{2R_1R_2 \sin\beta}{R_1 + R_2} \tag{45}$$

ie $$\frac{L^2 - L^2 + 2Ld - d^2}{L} = \frac{2R_1R_2 \sin\beta}{R_1 + R_2}$$

ie $$2Ld - d^2 = \frac{2LR_1R_2 \sin\beta}{R_1 + R_2}$$

ie $$d^2 - 2Ld + \frac{2LR_1R_2 \sin\beta}{R_1 + R_2} = 0 \tag{46}$$

Now solve for the overhang d as a quadratic, where

$a = 1$
$b = -2L$
$c = \dfrac{2LR_1R_2 \sin\beta}{R_1 + R_2}$

so $$d = \frac{2L \pm \left[4L^2 - \frac{4.1.2L R_1R_2 \sin\beta}{(R_1+R_2)}\right]^{\frac{1}{2}}}{2} \qquad (47)$$

$$= \frac{2L \pm 2\left[L^2 - \frac{2L R_1R_2 \sin\beta}{(R_1+R_2)}\right]^{\frac{1}{2}}}{2}$$

ie $$d = L \pm \left[L^2 - \frac{2L R_1R_2 \sin\beta}{(R_1+R_2)}\right]^{\frac{1}{2}} \qquad (48)$$

For a real solution, take the negative sign.

ie $$d = L - \left[L^2 - \frac{2L R_1R_2 \sin\beta}{(R_1+R_2)}\right]^{\frac{1}{2}} \qquad (49)$$

Now substitute (36) into (49):

ie $$d = L - \left[L^2 - \frac{2L R_1R_2}{R_1+R_2} \cdot \frac{R_1R_2(R_1+R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}\right]^{\frac{1}{2}}$$

ie $$d = L - \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2}\right]^{\frac{1}{2}} \qquad (50)$$

EQN (50) is the standard expression for overhang.

Now, as $d = L - M$, then from (50):

$$M = \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2}\right]^{\frac{1}{2}} \qquad (51)$$

Note that the right hand term in the brackets of (51) is Ra^2, so

$$M = \left[L^2 - Ra^2\right]^{\frac{1}{2}} \qquad (52)$$

which is identical to Löfgren's EQN (39). GRAEME F. DENNES

DERIVATION OF Ra^2 AND M
LÖFGREN'S EQN (40) AND (39)

The radius Ra is the radius at the minimum of the tracking error function,
where tracking error $\phi = \alpha - \beta$.

To determine Ra, diffentiate the tracking error function, equate it to zero, and solve for R = Ra.

From EQN (17),

$$\phi = \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta} \text{ radians}$$

$$= \left[\frac{R^2}{2LR} + \frac{2Ld}{2LR} - \frac{d^2}{2LR} - \sin\beta\right]\frac{1}{\cos\beta}$$

$$= \frac{R}{2L\cos\beta} + \frac{d}{R\cos\beta} - \frac{d^2}{2LR\cos\beta} - \frac{\sin\beta}{\cos\beta} \qquad (53)$$

Now,

$$\frac{d}{dR} = \frac{1}{2L\cos\beta} - \frac{d}{R^2\cos\beta} + \frac{d^2}{2LR^2\cos\beta} = 0 \qquad (54)$$

Multiply by $2LR^2\cos\beta$:

$$= R^2 - 2Ld + d^2 = 0 \qquad (55)$$

$$\therefore \quad R^2 = 2Ld - d^2$$

$$\text{ie} \quad Ra^2 = 2Ld - d^2 \qquad (56)$$

Now substitute the expression (50) for d :

ie $$Ra^2 = 2L\left[L - \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2+R_1R_2}\right]^{\frac{1}{2}}\right]$$

$$- \left[L - \left[L^2 - \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2+R_1R_2}\right]^{\frac{1}{2}}\right]^2 \qquad (57)$$

To simplify the notation,

Let $$F = \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2+R_1R_2} \qquad (58)$$

So $$Ra^2 = 2L^2 - 2L\left[L^2 - F\right]^{\frac{1}{2}}$$

$$- \left[L^2 - 2L\left[L^2 - F\right]^{\frac{1}{2}} + \left[L^2 - F\right]\right] \qquad (59)$$

$$= 2L^2 - 2L\left[L^2 - F\right]^{\frac{1}{2}}$$

$$- L^2 + 2L\left[L^2 - F\right]^{\frac{1}{2}} - \left[L^2 - F\right]$$

$$= 2L^2 - L^2 - \left[L^2 - F\right]$$

$$= L^2 - \left[L^2 - F\right]$$

$$= L^2 - L^2 + F$$

$$Ra^2 = F \qquad (60)$$

So $$Ra^2 = \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2+R_1R_2} \qquad (61)$$

$$\text{ie} \quad Ra^2 = \frac{8 R_1^2 R_2^2}{(R_1 + R_2)^2 + 4 R_1 R_2} \tag{62}$$

Equation (62) is identical to Lötgren's EQN (40).

Now, as $Ra^2 = 2Ld - d^2,$ (63)

and $d = L - M,$ (64)

then $Ra^2 = 2L(L-M) - (L-m)^2$

$= 2L^2 - 2LM - L^2 + 2LM - M^2$

ie $Ra^2 = L^2 - M^2$ (65)

then $M^2 = L^2 - Ra^2$

and $M = \left[L^2 - Ra^2\right]^{\frac{1}{2}}$ (66)

EQN (66) is identical to Lötgren's EQN (39).

DERIVATION OF LINEAR OFFSET LOFGREN'S EQNS (41) & (42).

From Fig. 1, $\sin\beta = \dfrac{\rho}{L}$ (67)

This is equivalent to Löfgren's EQN (41), because γ_0 and β are complementary, where $\gamma_0 = \gamma$ when $\phi' = 0$.

So $\rho = L \sin\beta$ (68)

$$= L \cdot \frac{R_1 R_2 (R_1 + R_2)}{L\left[\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2\right]}$$

ie $$\rho = \frac{R_1 R_2 (R_1 + R_2)}{\frac{1}{4}(R_1 + R_2)^2 + R_1 R_2} \qquad (69)$$

This is identical to Löfgren's EQN (42).

DERIVATION OF LÖFGREN'S MAXIMUM WTE EQUATION FOR THE 'LÖFGREN A' SOLUTION [EQN (43)]

Starting with the WTE part of Löfgren's EQN (45):

$$WTE = \frac{1}{[L^2 - \rho^2]^{\frac{1}{2}}} \cdot \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2}\right] \qquad (70)$$

For the 'Löfgren A' solution, we have three "maxima" of the WTE: at R_1, R_W & R_2.

In (70), let $R = R_W$, which is one of the WTE "maxima" radii.
Also, let $Ra^2 = R_W L \sin\beta$ from EQN D5 on page 9-3. Thus:

Now WTE max. = − WTE at R_W

$$= - \frac{1}{[L^2 - \rho^2]^{\frac{1}{2}}} \cdot \left[\frac{1}{2} - \frac{\rho}{R_W} + \frac{R_W L \sin\beta}{2R_W^2}\right] \qquad (71)$$

$$= - \quad '' \quad \cdot \left[\frac{1}{2} - \frac{\rho}{R_W} + \frac{L\sin\beta}{2R_W}\right]$$

$$= - \quad '' \quad \cdot \left[\frac{1}{2} - \frac{2\rho}{2R_W} + \frac{\rho}{2R_W}\right]$$

$$= - \quad '' \quad \cdot \left[\frac{1}{2} - \frac{\rho}{2R_W}\right]$$

$$= - \quad '' \quad \cdot \frac{1}{2}\left[1 - \frac{\rho}{R_W}\right]$$

$$= \quad - \frac{1}{2\left[L^2 - \rho^2\right]^{\frac{1}{2}}} \cdot \left[1 - \frac{\rho}{RW}\right] \qquad (72)$$

$$= \quad \frac{1}{2\left[L^2 - \rho^2\right]^{\frac{1}{2}}} \cdot \left[\frac{\rho}{RW} - 1\right] \qquad (73)$$

$$= \quad \frac{\frac{\rho}{RW} - 1}{2\left[L^2 - \rho^2\right]^{\frac{1}{2}}} \qquad (74)$$

The equation above is identical to Löfgren's EQN (43).

DERIVATION OF LÖFGREN'S EQN (35) FOR WTE AT R1 AND R2

From EQN (22):

$$WTE = \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta}$$

For the 'Löfgren A' solution, the WTE at R1 and R2 are equal in magnitude and sign, so for $R = R_1$:

$$WTE \text{ at } R_1 = \left[\frac{R_1^2 + L^2 - M^2}{2LR_1} - \sin\beta\right]\frac{1}{R_1\cos\beta} \quad (75)$$

From (36C):

$$\sin\beta = \frac{L^2 - M^2}{2L}\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$= \frac{L^2 - M^2}{2L} \cdot \frac{R_1 + R_2}{R_1 R_2} \quad (76)$$

By substituting (76) into (75) for $\sin\beta$, Löfgren sought to simplify (75), as follows:

$$WTE \text{ at } R_1 = \left[\frac{R_1^2 + L^2 - M^2}{2LR_1} - \frac{L^2 - M^2}{2L} \cdot \frac{R_1 + R_2}{R_1 R_2}\right]\frac{1}{R_1\cos\beta} \quad (77)$$

$$= \left[\frac{R_1^2 + L^2 - M^2}{2LR_1^2} - \frac{(L^2 - M^2)(R_1 + R_2)}{2LR_1^2 R_2}\right]\frac{1}{\cos\beta}$$

Multiply first term by $\frac{R_2}{R_2}$:

$$= \left[\frac{(R_1^2 + L^2 - M^2)R_2}{2LR_1^2R_2} - \frac{(L^2 - M^2)(R_1 + R_2)}{2LR_1^2R_2} \right] \frac{1}{\cos\beta} \qquad (78)$$

$$= \left[\frac{R_1^2R_2 - L^2R_1 + M^2R_1}{2LR_1^2R_2} \right] \frac{1}{\cos\beta}$$

$$= \left[\frac{R_1R_2 - L^2 + M^2}{2LR_1R_2} \right] \frac{1}{\cos\beta}$$

So WTE at R_1

$$= \frac{R_1R_2 - L^2 + M^2}{2LR_1R_2\cos\beta}$$

As the WTE at R_2 = WTE at R_1, then

$$\text{WTE at } R_1, R_2 = \frac{R_1R_2 - L^2 + M^2}{2LR_1R_2\cos\beta} \qquad (79)$$

(79) is equivalent to Löfgren's EQN (35).

DERIVATION OF LÖFGREN'S EQN (38) WTE AT R_W

From (22):

$$\text{WTE} = \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (80)$$

$$= \left[\frac{R^2 + L^2 - M^2}{2LR^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta} \qquad (81)$$

For the 'Löfgren A' solution, from (43):

$$R_W = \frac{2R_1R_2}{R_1 + R_2} \qquad (82)$$

so at $R = R_W$,

$$\text{WTE} = \left[\frac{R_W^2 + L^2 - M^2}{2LR_W^2} - \frac{\sin\beta}{R_W}\right]\frac{1}{\cos\beta} \qquad (83)$$

Taking the first term in brackets:

$$\frac{R_W^2 + L^2 - M^2}{2LR_W^2} = \frac{\left[\left(\frac{2R_1R_2}{R_1+R_2}\right)^2 + L^2 - M^2\right](R_1+R_2)^2}{2L(2R_1R_2)^2}$$

$$= \left[\frac{(2R_1R_2)^2 + (L^2 - M^2)(R_1+R_2)^2}{2L(R_1+R_2)^2\cdot(2R_1R_2)^2}\right](R_1+R_2)^2$$

$$= \frac{(2R_1R_2)^2 + (L^2-M^2)(R_1+R_2)^2}{2L(2R_1R_2)^2}$$

$$= \frac{(2R_1R_2)^2 + (L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2} \qquad (84)$$

Taking the second term in the brackets:

From (36C):

$$\sin\beta = \frac{L^2-M^2}{2L}\left[\frac{1}{R_1}+\frac{1}{R_2}\right] \qquad (85)$$

$$= \frac{L^2-M^2}{2L}\cdot\frac{R_1+R_2}{R_1R_2} \qquad (86)$$

so $$\frac{\sin\beta}{R_W} = \frac{L^2-M^2}{2L}\cdot\frac{R_1+R_2}{R_1R_2}\cdot\frac{R_1+R_2}{2R_1R_2}$$

$$= \frac{(L^2-M^2)(R_1+R_2)^2}{4LR_1^2R_2^2} \qquad (87)$$

so WTE at R_W is:

$$\left[\frac{(2R_1R_2)^2+(L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2} - \frac{(L^2-M^2)(R_1+R_2)^2}{4LR_1^2R_2^2}\right]\frac{1}{\cos\beta} \qquad (88)$$

$$= \left[\frac{(2R_1R_2)^2+(L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2} - \frac{2(L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{4R_1^2R_2^2+(L^2-M^2)(R_1+R_2)^2 - 2(L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{4R_1^2R_2^2-(L^2-M^2)(R_1+R_2)^2}{8LR_1^2R_2^2}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{4R_1^2R_2^2 - (L^2-M^2)(R_1+R_2)^2}{4R_1^2R_2^2} \right] \frac{1}{2L\cos\beta}$$

$$= \left[1 - \frac{(L^2-M^2)(R_1+R_2)^2}{4R_1^2R_2^2} \right] \frac{1}{2L\cos\beta}$$

$$= \left[1 - \frac{L^2-M^2}{4} \cdot \frac{(R_1+R_2)^2}{(R_1R_2)^2} \right] \frac{1}{2L\cos\beta}$$

$$= \left[1 - \frac{L^2-M^2}{4} \cdot \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2 \right] \frac{1}{2L\cos\beta} \qquad (89)$$

$$\equiv -\left[\frac{L^2-M^2}{4} \cdot \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2 - 1 \right] \frac{1}{2L\cos\beta} \qquad (90)$$

EQN (90) is equivalent to Löfgren's EQN (38).

DETAILED DERIVATION OF LÖFGREN'S 'LÖFGREN B' ALIGNMENT EQUATIONS

This Section presents the detailed derivation of the alignment equations presented by Löfgren for the 'Löfgren B' alignment.

1. EQN (45) (Part of)	Weighted tracking error
2. EQN (46)	Tracking distortion
3. EQN (47)	Minimum angular lateral tracking error radius R_a
4. EQN (48)	Optimum overhang

DERIVATION OF LÖFGREN'S APPROXIMATE WTE EXPRESSION (WTE PART OF LÖFGREN'S EQN (45))

Löfgren's EQN (33) is an excellent approximation to the WTE. However, like the exact expression for WTE at EQN J on page S9-6, it uses trigonometric functions, making it difficult to use in distortion calculations when integration is required.

Löfgren developed the WTE part of his EQN (45) to enable it to be used in distortion calculations.

This section commences with Löfgren's EQN (33), then derives an equivalent expression for the WTE which does not use trigonometric functions. Löfgren then uses this expression as a part of his EQN (45). His EQN (45) is then equivalent to his EQN (22), indicating the level of second harmonic distortion.

Löfgren presents the following expressions:

EQN (39): $$M = \left(L^2 - Ra^2\right)^{\frac{1}{2}} \qquad (1)$$

EQN (41): $$\cos \gamma_0 = \frac{\rho}{L} \quad \left(= \sin \beta\right) \qquad (2)$$

Starting with Löfgren's EQN (33):

$$\text{WTE} = \frac{1}{R \sin \gamma_0} \cdot \left[\frac{R^2 + L^2 - M^2}{2LR} - \cos \gamma_0\right] \qquad (3)$$

$$\equiv \frac{1}{R \cos \beta} \cdot \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin \beta\right] \qquad (4)$$

$$= \quad " \quad \cdot \left[\frac{R^2 + L^2 - (L^2 - Ra^2)}{2LR} - \sin \beta\right]$$

$$= \quad " \quad \cdot \left[\frac{R^2 + Ra^2}{2LR} - \frac{\rho}{L}\right]$$

$$= \quad " \quad \cdot \left[\frac{R^2}{2LR} + \frac{Ra^2}{2LR} - \frac{\rho}{L}\right]$$

$$= \frac{1}{R\cos\beta} \cdot \left[\frac{R}{2L} - \frac{\rho}{L} + \frac{Ra^2}{2LR}\right]$$

$$= \frac{1}{R\cos\beta} \cdot \frac{R}{L} \cdot \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2}\right]$$

$$= \frac{1}{L\cos\beta} \cdot \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2}\right]$$

$$= \frac{1}{\left[L^2\cos^2\beta\right]^{\frac{1}{2}}} \cdot \quad ''$$

But $\cos^2 x = 1 - \sin^2 x$, so

$$= \frac{1}{\left[L^2\left(1-\sin^2\beta\right)\right]^{\frac{1}{2}}} \cdot \quad ''$$

$$= \frac{1}{\left[L^2 - L^2\sin^2\beta\right]^{\frac{1}{2}}} \cdot \quad ''$$

$$= \frac{1}{\left[L^2 - (L\sin\beta)^2\right]^{\frac{1}{2}}} \cdot \quad ''$$

$$\text{WTE} = \frac{1}{\left[L^2 - \rho^2\right]^{\frac{1}{2}}} \cdot \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2}\right] \quad (5)$$

radians per unit length.

This is the WTE part of Löfgren's EQN (45), which is equivalent to his EQN (33), both of which are approximations to the exact WTE equation at EQN (J) on page S9-6 of this analysis. Most importantly, the removal of all trigonometric functions simplifies its use in distortion calculations, where integration is required, facilitating the derivation of the 'Löfgren B' optimum equations, as undertaken by Löfgren.

DERIVATION OF LÖFGREN'S DISTORTION EXPRESSION [LÖFGREN'S EQN (46)]

Löfgren's distortion expression at EQN (46) is derived from his EQN (45), as follows. EQN (46) enables the mean distortion and the Root Mean Square (RMS) distortion to be calculated for the distortion created by tracking error.

Löfgren argues that a suitable quantitative measure for the tracking distortion is the RMS distortion figure, shown at his EQN (44):

$$K_{eff} = \left[\frac{1}{R_2 - R_1} \int_{R_1}^{R_2} K^2 \, dR \right]^{\frac{1}{2}} \qquad (6)$$

where Keff is the effective or RMS distortion.

To determine the RMS distortion, we must:

1. Define a distortion parameter K.
2. Integrate the square of the distortion parameter.
3. Calculate the mean distortion by dividing the integral by the interval of integration (or base).
4. Calculate the RMS distortion by taking the square root of the mean value.

At EQN (45), Löfgren presents the distortion parameter K:

$$K \approx \frac{V}{\Omega} \cdot \frac{\phi}{R} \qquad (7)$$

where V = peak recorded velocity
Ω = record angular velocity
$\frac{\phi}{R}$ = WTE at radius R

Substituting the WTE part of Löfgren's EQN (45):

ie $$K \approx \frac{V}{\Omega} \cdot \frac{1}{(L^2 - \rho^2)^{\frac{1}{2}}} \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} \right] \qquad (8)$$

so $$K^2 \approx \frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - \rho^2} \left[\frac{1}{2} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} \right]^2 \qquad (9)$$

Now expand the brackets:

$$K^2 \approx \frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - \rho^2} \left[\frac{1}{4} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} + \frac{\rho^2}{R^2} - \frac{\rho Ra^2}{R^3} + \frac{Ra^4}{4R^4} \right] \qquad (10)$$

Now, the mean distortion is :

$$\frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - \rho^2} \cdot \frac{1}{R_2 - R_1} \int_{R_1}^{R_2} \frac{1}{4} - \frac{\rho}{R} + \frac{Ra^2}{2R^2} + \frac{\rho^2}{R^2} - \frac{\rho Ra^2}{R^3} + \frac{Ra^4}{4R^4} \, dR \qquad (11)$$

After completing the integration, the mean distortion is :

$$\frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - \rho^2} \cdot \frac{1}{R_2 - R_1} \left| \frac{R}{4} - \rho \ln R - \frac{Ra^2}{2R} - \frac{\rho^2}{R} + \frac{\rho Ra^2}{2R^2} - \frac{Ra^4}{12R^3} \right|_{R_1}^{R_2} \qquad (12)$$

After the limits of integration are applied, the mean distortion is :

$$\frac{V^2}{\Omega^2} \cdot \frac{1}{L^2 - \rho^2} \cdot \frac{1}{R_2 - R_1} \left[\frac{R_2 - R_1}{4} - \rho \ln \frac{R_2}{R_1} + \frac{Ra^2}{2} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right.$$

$$+ \rho^2 \left(\frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{\rho Ra^2}{2} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right)$$

$$\left. + \frac{Ra^4}{12} \left(\frac{1}{R_1^3} - \frac{1}{R_2^3} \right) \right] \qquad (13)$$

This is identical to Löfgren's EQN (46).

EXAMPLE OF RMS DISTORTION CALCULATION

1. Use the previous expression to calculate the mean distortion (EQN 13).
2. The RMS distortion is calculated by taking the square root of the mean distortion.

Use the following data to calculate the mean and RMS distortion figures.

$L = 250$ mm
$V = 100$ mm/sec peak
$\Omega = 3 \cdot 49$ radians/sec (for the LP)
$Ra^2 = 7744$ mm^2
$\rho = 93 \cdot 6516$ mm
$\beta = 0 \cdot 38397$ radians ($22°$)
$d = 16$ mm
$R_1 = 60 \cdot 325$ mm
$R_2 = 146 \cdot 05$ mm

<u>The mean distortion is</u> :

$$1 \cdot 7818 \times 10^{-4} \left[21 \cdot 43125 - 82 \cdot 80701 + 37 \cdot 674 + 85 \cdot 337 - 82 \cdot 6451 + 21 \cdot 16 \right]$$

$$= 2 \cdot 689 \times 10^{-5} = 0 \cdot 00002689$$

<u>The RMS distortion is</u> :

$$\left(2 \cdot 689 \times 10^{-5}\right)^{\frac{1}{2}} = 5 \cdot 1865 \times 10^{-3}$$

$$= 0 \cdot 0051865$$

$$= 0 \cdot 51865 \%$$

<u>NOTE : RIAA CORRECTION NOT INCLUDED</u>

DERIVATION OF LÖFGREN'S 'LÖFGREN B' EQUATIONS EQN (47) and (48)

Background

The 'Löfgren B' alignment is a compromise alignment used when the *offset angle is fixed and non-optimum*, and is based on adjusting the overhang to achieve minimum LMS (or RMS) tracking distortion. Löfgren develops an overhang equation at EQNs (47) and (48) for use in this situation. (When the offset angle is adjustable, the 'Löfgren A' solution is used!) Löfgren argued that the largest distortion risk occurs when the overhang is not correctly set for the offset angle, while the offset angle itself is not so critical. As a consequence, this allowed for a fixed, non-optimum offset angle to be accommodated through minimising the LMS (or RMS) distortion.

The derivation of the 'Löfgren B' overhang equation will now be undertaken.

'Löfgren B' Overhang Equation Derivation

1. Starting with his EQN (33), an excellent approximation to the WTE, Löfgren derived the WTE part (the δ/R part) of his EQN (45). It intentionally contains no trigonometric functions, simplifying its further analysis. Otherwise it is identical to EQN (33), apart from its reformulation. The derivation of the WTE part of EQN (45) is shown elsewhere in this analysis.

   EQN (45) is equivalent to his EQN (22), which indicates the level of second harmonic distortion due to lateral tracking error. (The derivation of EQN (22) is also shown elsewhere in this analysis). Löfgren developed EQN (45) specifically as a distortion indicator or distortion factor, K, to enable its further analysis.

2. The distortion factor K (EQN (45)) is then squared (K^2), then integrated. The result, K^2_{eff}, is shown at EQN (46), and gives the mean distortion figure. The RMS distortion is calculated by taking the square root of the mean distortion.

To determine an expression for the overhang, we observe that $M = (L^2 - Ra^2)^{1/2}$ from Löfgren's EQN (39), where L is the arm length and M is the mounting distance, from which the overhang d = L - M is determined. We observe that the mean distortion, K^2_{eff}, given by EQN (46), is a function of Ra^2 (amongst others), so a change in Ra^2 will change the mean distortion. Therefore, the mean distortion given by EQN (46) will be at its minimum value if we can find the value for Ra^2 to achieve this. This Ra^2 value is then used to calculate the overhang. When this overhang is implemented with the non-optimum offset angle, we will achieve the 'Löfgren B' alignment condition of minimum LMS (or RMS) distortion.

Minimising the Mean (and RMS) Distortion

Firstly, we need to determine an expression for Ra^2 which will minimise the mean distortion given by Löfgren's EQN (46). Let's look at EQN (46) in detail (Löfgren chose to combine the third and fourth terms in the square brackets).

Löfgren's EQN (46) is as follows:

Mean distortion $\left(K^2_{eff}\right)$ =

$$\frac{V^2}{\Omega^2} \cdot \frac{1}{L^2-\rho^2} \cdot \frac{1}{R_2-R_1} \left[\frac{R_2-R_1}{4} - \rho \ln\frac{R_2}{R_1} + \frac{Ra^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \right. \quad (14)$$

$$\left. + \rho^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) - \rho\frac{Ra^2}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) + \frac{Ra^4}{12}\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right) \right]$$

To find the expression for Ra^2 which will minimise EQN (46), we will firstly differentiate EQN (46), equate the derivative to zero, then solve for Ra^2. When that expression for Ra^2 is put back into EQN (46), the resulting RMS distortion is at its minimum value.

We observe that EQN (46) consists of two parts, namely the part before the square brackets, and the part within the square brackets. Looking at the former, we note it is a non-zero constant comprised of several input variables, so will cancel out when we equate the derivative to zero after the differentiation step. Looking at the part within the square brackets, it consists of six terms, three being constants, which will also fall out during the process, and three containing the Ra^2 term which we are interested in.

For our purposes here, we will now differentiate EQN (46) with respect to Ra^2, and consider the term Ra^4 as being $(Ra^2)^2$.

$$\frac{d}{dRa^2}\,\frac{Ra^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{d}{dRa^2}\,-\rho\frac{Ra^2}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) = -\frac{\rho}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) \quad (15)$$

$$\frac{d}{dRa^2}\,\frac{Ra^4}{12}\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right) = \frac{Ra^2}{6}\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right)$$

We now equate the derivative to zero, and solve for Ra^2.

$$\frac{Ra^2}{6}\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right)-\frac{\rho}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)+\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=0 \quad (16)$$

$$\text{ie } Ra^2=\frac{\frac{\rho}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)-\frac{1}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}{\frac{1}{6}\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right)} \quad (17)$$

$$=\frac{\rho\left(\frac{R_2^2-R_1^2}{R_1^2R_2^2}\right)-\left(\frac{R_2-R_1}{R_1R_2}\right)}{\frac{1}{3}\left(\frac{R_2^3-R_1^3}{R_1^3R_2^3}\right)}$$

$$=\frac{3R_1^3R_2^3\left[\rho\left(\frac{R_2^2-R_1^2}{R_1^2R_2^2}\right)-\left(\frac{R_2-R_1}{R_1R_2}\right)\right]}{R_2^3-R_1^3}$$

$$=\frac{3R_1^3R_2^3\left[\rho\left[\frac{(R_2+R_1)(R_2-R_1)}{R_1^2R_2^2}\right]-\left(\frac{R_2-R_1}{R_1R_2}\right)\right]}{(R_2-R_1)(R_2^2+R_1R_2+R_1^2)}$$

$\div (R_2-R_1)$:

$$=\frac{3R_1^3R_2^3\left[\rho\left[\frac{(R_2+R_1)}{R_1^2R_2^2}\right]-\left(\frac{1}{R_1R_2}\right)\right]}{R_2^2+R_1R_2+R_1^2}$$

Multiply second term by $\frac{R_1 R_2}{R_1 R_2}$:

ie $$Ra^2 = \frac{3R_1^3 R_2^3 \left[\rho \left(\frac{R_2 + R_1}{R_1^2 R_2^2} \right) - \frac{R_1 R_2}{R_1^2 R_2^2} \right]}{R_2^2 + R_1 R_2 + R_1^2}$$

$$= \frac{\frac{3R_1^3 R_2^3}{R_1^2 R_2^2} \left[\rho \left(R_2 + R_1 \right) - R_1 R_2 \right]}{R_2^2 + R_1 R_2 + R_1^2}$$

ie $$Ra^2 = \frac{3R_1 R_2 \left[\rho \left(R_1 + R_2 \right) - R_1 R_2 \right]}{R_2^2 + R_1 R_2 + R_1^2} \qquad (18)$$

When we substitute the above expression for Ra^2 into the mean distortion equation at EQN (46), the resulting mean distortion will be at its minimum value, as will be the RMS distortion.

The equation above is identical to Löfgren's EQN (48).

Calculation of Overhang

We now make use of the calculated Ra^2 expression to determine the overhang required to achieve the minimum LMS distortion condition (the 'Löfgren B' alignment) for when the offset angle is fixed and non-optimum.

The overhang expression is then:

$$d = L - \left(L^2 - Ra^2\right)^{\frac{1}{2}}$$

$$= L - \left[L^2 - \frac{3R_1R_2\left[\rho\left(R_1 + R_2\right) - R_1R_2\right]}{R_2^2 + R_1R_2 + R_1^2}\right]^{\frac{1}{2}} \qquad (19)$$

This completes the derivation of the 'Löfgren B' overhang expression which results in the LMS (and RMS) distortion being minimised, per the 'Löfgren B' alignment, for when the offset angle is fixed and non-optimum.

The equation above is identical to Löfgren's EQN (47).

DETAILED DERIVATION OF ADDITIONAL 'LÖFGREN A' EQUATIONS

This Section presents the derivation of additional equations.

1. Null radii R_{01} and R_{02}
2. (True) null radii
3. The derivative of the approximate weighted tracking error
4. The (true) weighted tracking error
5. The derivative of the (true) weighted tracking error
6. Löfgren's R_a^2 term is the product of the null radii
7. Null radii average
8. Radius R_2 in terms of null radii
9. Radius R_1 in terms of null radii
10. Stevenson's R_W from Löfgren's R_W
11. Kessler/Pisha EQN (3)
12. Kessler/Pisha EQN (4)
13. Baerwald's EQN (13) is identical to Löfgren's EQN (2)
14. Baerwald's EQN (14) is identical to Löfgren's EQN (33)
15. (Part of) Baerwald's EQN (16b) is identical to Löfgren's EQN (40)
16. Baerwald's EQN (10) is identical to Löfgren's EQN (22)
17. (Part of) Baerwald's EQN (16e) is identical to Löfgren's EQN (37)
18. Seagrave's EQN (18) WTE at Radius R
19. The Equivalence of Seagrave's EQN (18) and Löfgren's EQN (33)
20. Seagrave's EQN (19) WTE at Rw
21. Bauer's Tracking Angle Equation EQN (5)
22. The Equivalence of Bauer's (1949) EQN (2) and Seagrave's EQN (26)
23. "The Best" Non-Optimum 'Löfgren A' Design Equations
24. 'Löfgren A' and 'Löfgren B' Zero Radii Calculations

DERIVATION OF THE 'LÖFGREN A' NULL RADII

The zero tracking error radii or null radii are the radius values at which the WTE is zero, ie, are the radius values at the two points where the WTE curve crosses the zero WTE axis.

Although Löfgren did not discuss null radii, he would certainly have known about them, and that their product is Ra^2.

We'll now derive the equations for the null radii.

Consider the WTE equation at EQN (21) on page S10-16:

$$WTE = \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (1)$$

radians per unit length.

$$\text{where } 2Ld - d^2 = Ra^2 = \frac{2R_1^2 R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \qquad (2)$$

$$\text{and } \sin\beta = \frac{R_1R_2(R_1+R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \qquad (3)$$

To find the two null radii, ie the two roots, equate the WTE equation to zero, and solve for R, the two null radii values.

We proceed as follows.

Expand the WTE equation as follows.

$$WTE = \left[\frac{R^2}{2LR^2} + \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} \cdot \frac{1}{2LR^2} - \frac{R_1R_2(R_1+R_2)}{L\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \cdot \frac{1}{R}\right]\frac{1}{\cos\beta} \qquad (4)$$

Now equate to zero. The $\cos\beta$ term drops out.

$$\text{so } WTE = \left[\frac{1}{2L} + \frac{2R_1^2R_2^2}{\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]LR^2} - \frac{R_1R_2(R_1+R_2)}{LR\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]}\right] = 0 \qquad (5)$$

$$\text{ie } \frac{1}{2L} + \frac{R_1^2R_2^2}{LR^2\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} - \frac{R_1R_2(R_1+R_2)}{LR\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} = 0$$

Multiply by $2LR^2$:

$$\text{ie } R^2 - \frac{R.2R_1R_2(R_1+R_2)}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} + \frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2 + R_1R_2} = 0 \qquad (6)$$

Multiply by $\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]$:

$$\text{ie } R^2\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right] - R.2R_1R_2(R_1+R_2) + 2R_1^2R_2^2 = 0 \qquad (7)$$

We now have a quadratic, so solve for R.

where $a = \frac{1}{4}(R_1+R_2)^2 + R_1R_2$; $b = -2R_1R_2(R_1+R_2)$;

$c = 2R_1^2R_2^2$

so $$R = \frac{-b \pm \left(b^2 - 4ac\right)^{\frac{1}{2}}}{2a}$$

$$= \frac{2R_1R_2(R_1+R_2) \pm \left[4R_1^2R_2^2(R_1+R_2)^2 - 4\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]2R_1^2R_2^2\right]^{\frac{1}{2}}}{2\left[\frac{1}{4}(R_1+R_2)^2 + R_1R_2\right]} \quad (8)$$

$$= \frac{2R_1R_2(R_1+R_2) \pm \left[2R_1^4R_2^2 - 4R_1^3R_2^3 + 2R_1^2R_2^4\right]^{\frac{1}{2}}}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2}$$

$$= \frac{2R_1R_2(R_1+R_2) \pm \left[2R_1^2R_2^2(R_1^2 - 2R_1R_2 + R_2^2)\right]^{\frac{1}{2}}}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2}$$

$$= \frac{2R_1R_2(R_1+R_2) \pm \sqrt{2}R_1R_2(R_1-R_2)}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2}$$

$$= \frac{2R_1^2R_2 + 2R_1R_2^2 \pm \left[\sqrt{2}R_1^2R_2 - \sqrt{2}R_1R_2^2\right]}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2} \quad (9)$$

Continue to next page.

The Solution For Ro1

From EQN (9), if we take the + sign, it will provide the expression for Ro1.

ie $$R_{o1} = \frac{2R_1^2R_2 + 2R_1R_2^2 + \sqrt{2}R_1^2R_2 - \sqrt{2}R_1R_2^2}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2} \qquad (10)$$

Multiply by $\frac{2}{2}$:

$$R_{o1} = \frac{4R_1^2R_2 + 4R_1R_2^2 + 2\sqrt{2}R_1^2R_2 - 2\sqrt{2}R_1R_2^2}{(R_1+R_2)^2 + 4R_1R_2}$$

$$= \frac{2R_1R_2[2R_1 + 2R_2 + \sqrt{2}R_1 - \sqrt{2}R_2]}{R_1^2 + R_2^2 + 6R_1R_2}$$

ie $$R_{o1} = \frac{2R_1R_2[R_1(2+\sqrt{2}) + R_2(2-\sqrt{2})]}{R_1^2 + R_2^2 + 6R_1R_2} \qquad (11)$$

EQN (11) can be further reduced by division with itself, as follows.

The numerator term in brackets is divided by itself, with a result of unity. We then divide the denominator by the bracketed numerator term using algebraic long division, as laid out as follows:

$$
\begin{array}{r|llll}
 & \frac{R_1}{2+\sqrt{2}} & + \frac{R_2}{2-\sqrt{2}} & & \\
\hline
R_1(2+\sqrt{2}) + R_2(2-\sqrt{2}) & R_1^2 & + \; R_2^2 & + \; 6R_1R_2 & \\
 & -R_1^2 & & - (3-2\sqrt{2})R_1R_2 & \\
\cline{2-4}
 & 0 & + \; R_2^2 & + (3+2\sqrt{2})R_1R_2 & \\
 & & - R_2^2 & - (3+2\sqrt{2})R_1R_2 & \\
\cline{3-4}
 & & 0 & \quad 0 &
\end{array}
$$

EQN (11) has now been reduced to:

$$R_{01} = \frac{2R_1R_2\left[\quad 1 \quad\right]}{\frac{R_1}{2+\sqrt{2}} + \frac{R_2}{2-\sqrt{2}}} \qquad (12)$$

The numerator has been reduced to: $2R_1R_2$. The denominator can be rationalised by taking each term in turn, as follows.

$$\frac{R_1}{2+\sqrt{2}} = \frac{R_1}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{R_1(2-\sqrt{2})}{2} = R_1\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$= R_1\left(1 - \frac{1}{\sqrt{2}}\right) \qquad (13)$$

Now take the second term:

$$\frac{R_2}{2-\sqrt{2}} = \frac{R_2}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{R_2(2+\sqrt{2})}{2} = R_2\left(1 + \frac{\sqrt{2}}{2}\right)$$

$$= R_2\left(1 + \frac{1}{\sqrt{2}}\right) \qquad (14)$$

Applying the new denominator terms, we have:

$$R_{01} = \frac{2R_1R_2}{R_1\left(1-\frac{1}{\sqrt{2}}\right) + R_2\left(1+\frac{1}{\sqrt{2}}\right)} \qquad (15)$$

This completes the derivation of the equation for the null radius R_{01}.

This is also shown at Baerwalds' EQN (16e).

Now proceed to the derivation of null radius R_{02} on the next page.

R_{01} Long Division Worksheet

1. $$\frac{R_1}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} \cdot R_2(2-\sqrt{2}) = \frac{R_1R_2}{2} \cdot 2-\sqrt{2} \cdot 2-\sqrt{2}$$

$$= \frac{R_1R_2(6-4\sqrt{2})}{2} = R_1R_2(3-2\sqrt{2})$$

2. $$6-(3-2\sqrt{2}) = 6-3+2\sqrt{2} = 3+2\sqrt{2}$$

3. $$R_1(2+\sqrt{2}) \cdot \frac{R_2}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{R_1R_2(6+4\sqrt{2})}{2}$$

$$= R_1R_2(3+2\sqrt{2})$$

The Solution For Ro2

From EQN (9), if we take the − sign, it will provide the expression for Ro2.

$$\text{ie } R_{o2} = \frac{2R_1^2R_2 + 2R_1R_2^2 - \sqrt{2}R_1^2R_2 + \sqrt{2}R_1R_2^2}{\frac{1}{2}(R_1+R_2)^2 + 2R_1R_2} \qquad (16)$$

Multiply by $\frac{2}{2}$:

$$R_{o2} = \frac{4R_1^2R_2 + 4R_1R_2^2 - 2\sqrt{2}R_1^2R_2 + 2\sqrt{2}R_1R_2^2}{(R_1+R_2)^2 + 4R_1R_2}$$

$$= \frac{2R_1R_2\left[2R_1 + 2R_2 - \sqrt{2}R_1 + \sqrt{2}R_2\right]}{R_1^2 + R_2^2 + 6R_1R_2}$$

$$\text{ie } R_{o2} = \frac{2R_1R_2\left[R_1(2-\sqrt{2}) + R_2(2+\sqrt{2})\right]}{R_1^2 + R_2^2 + 6R_1R_2} \qquad (17)$$

EQN (17) can be further reduced by division with itself, as follows.

The numerator term in brackets is divided by itself, with a result of unity. We then divide the denominator by the bracketed term using algebraic long division, as follows.

$$\frac{R_1}{2-\sqrt{2}} + \frac{R_2}{2+\sqrt{2}}$$

$$R_1(2-\sqrt{2}) + R_2(2+\sqrt{2}) \overline{\Big)\; R_1^2 + R_2^2 + 6R_1R_2}$$

$$\begin{array}{rrr}
R_1^2 & + R_2^2 & + 6R_1R_2 \\
-R_1^2 & & -(3+2\sqrt{2})R_1R_2 \\
\hline
0 & + R_2^2 & +(3-2\sqrt{2})R_1R_2 \\
 & -R_2^2 & -(3-2\sqrt{2})R_1R_2 \\
\hline
 & 0 & 0
\end{array}$$

EQN (17) has now been reduced to:

$$R_{o2} = \frac{2R_1R_2[\quad 1 \quad]}{\dfrac{R_1}{2-\sqrt{2}} + \dfrac{R_2}{2+\sqrt{2}}} \qquad (18)$$

The numerator has been reduced to $2R_1R_2$. The denominator can be rationalised by taking each term in turn, as follows:

$$\frac{R_1}{2-\sqrt{2}} = \frac{R_1}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{R_1(2+\sqrt{2})}{2} = R_1\left(1+\frac{\sqrt{2}}{2}\right)$$

$$= R_1\left(1+\frac{1}{\sqrt{2}}\right) \qquad (19)$$

Now take the second term:

$$\frac{R_2}{2+\sqrt{2}} = \frac{R_2}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{R_2(2-\sqrt{2})}{2} = R_2\left(1-\frac{\sqrt{2}}{2}\right)$$

$$= R_2\left(1-\frac{1}{\sqrt{2}}\right) \qquad (20)$$

Applying the new denominator terms, we have:

$$R_{o2} = \frac{2R_1R_2}{R_1\left(1+\frac{1}{\sqrt{2}}\right) + R_2\left(1-\frac{1}{\sqrt{2}}\right)} \qquad (21)$$

This completes the derivation of the equation for the null radius R_{o2}.

This is also shown at Baerwald's EQN (16e).

R_{o2} Long Division Worksheet

1. $$R_1R_2 \cdot \frac{2+\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{R_1R_2(4+4\sqrt{2}+2)}{4-2}$$

$$= \frac{R_1R_2(6+4\sqrt{2})}{2} = R_1R_2(3+2\sqrt{2})$$

2. $$6-(3+2\sqrt{2}) = 6-3-2\sqrt{2} = 3-2\sqrt{2}$$

3. $$R_1 \cdot 2-\sqrt{2} \cdot \frac{R_2}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{R_1R_2(4-4\sqrt{2}+2)}{4-2}$$

$$= \frac{R_1R_2(6-4\sqrt{2})}{2} = R_1R_2(3-2\sqrt{2})$$

DERIVATION OF THE TRUE NULL RADII

The null radii are the roots of the WTE function shown at EQN J on page S9-6 ie the radius values where the WTE is zero.

Taking the WTE expression, we equate it to zero, and solve for the roots (radii).

Let $$\text{WTE} = \frac{\arcsin\left[\frac{R^2+L^2-M^2}{2LR}\right] - \beta}{R} = 0 \qquad (1)$$

ie $$\arcsin\left[\frac{R^2+L^2-M^2}{2LR}\right] - \beta = 0$$

ie $$\frac{R^2+L^2-M^2}{2LR} - \sin\beta = 0$$

ie $$R^2+L^2-M^2 - 2LR\sin\beta = 0$$

ie $$R^2 - 2L\sin\beta R + (L^2-M^2) = 0 \qquad (2)$$

Now solve for R, the null radii.

$$R = \frac{2L\sin\beta \pm \left[(2L\sin\beta)^2 - 4(L^2-M^2)\right]^{\frac{1}{2}}}{2} \qquad (3)$$

$$= L\sin\beta \pm \left[(L\sin\beta)^2 - L^2 + M^2\right]^{\frac{1}{2}} \qquad (4)$$

$$= \rho \pm \left[(L\sin\beta)^2 - (L^2-M^2)\right]^{\frac{1}{2}} \qquad (5)$$

$$= \rho \pm \left[\rho^2 - (2Ld-d^2)\right]^{\frac{1}{2}} \qquad (6)$$

$$= \rho \pm \left[\rho^2 - Ra^2\right]^{\frac{1}{2}} \qquad (7)$$

SOME OBSERVATIONS FROM EQNS. (4) AND (5)

1. The expression for the true null radii at EQN (4) has been derived directly from the WTE expression, which was derived using the Cosine Rule. NO simplifications or approximations have been used.

2. EQN (5) is derived from EQN (4) by observation.

3. The true null radii are functions of L, d, β only.

4. The values for L, d, β do not have to be optimum.

5. The inner and outer groove radii, R1 and R2, are not included in the solution for the true null radii.

6. The equations for the null radii as derived elsewhere in this analysis, and as presented by Baerwald at EQN C, page S9-3, are very good approximations, and are consistent with the 'Löfgren A' solution.

7. EQNS (4) and (5) show the true relationship between the null radii, the linear offset, and the WTE.

DERIVATIVE OF THE APPROXIMATE WTE EQUATION

From EQN H1 on page S9-5:

$$WTE = \left[\frac{R^2+L^2-M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (1)$$

$$= \frac{R^2}{2LR^2\cos\beta} + \frac{L^2-M^2}{2LR^2\cos\beta} - \frac{\sin\beta}{R\cos\beta}$$

$$= \frac{1}{2L\cos\beta} + \frac{L^2-M^2}{2LR^2\cos\beta} - \frac{\sin\beta}{R\cos\beta}, \quad \text{so}$$

$$\frac{d}{dR} = 0 - \frac{2(L^2-M^2)}{2LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta}$$

$$= -\frac{(L^2-m^2)}{LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta} \qquad (2)$$

We know that $(L^2-M^2) = 2Ld-d^2$ so

$$\frac{d}{dR} \text{ also} = -\frac{(2Ld-d^2)}{LR^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta} \qquad (3)$$

EQNS (2) and (3) are equivalent.

DERIVATION OF THE WTE EQUATION

The WTE equation is shown at EQN (J), page S9-6.

Refer to FIG. 1 on page S10-3, and EQN (4) on page S10-5.

Let Tracking angle $= \alpha$
Offset angle $= \beta$
Tracking error $= \alpha - \beta = \phi$

$$\text{So}\quad WTE = \frac{\alpha - \beta}{R} \tag{1}$$

$$\sin \alpha = \frac{R^2 + 2Ld - d^2}{2LR} \tag{2}$$

$$\therefore \quad \alpha = \arcsin\left(\frac{R^2 + 2Ld - d^2}{2LR}\right) \tag{3}$$

$$\text{So}\quad WTE = \frac{\arcsin\left(\frac{R^2 + 2Ld - d^2}{2LR}\right) - \beta}{R} \tag{4}$$

as shown at EQN (J2) on page S9-6.

Also,

$$WTE = \frac{\arcsin\left(\frac{R^2 + L^2 - m^2}{2LR}\right) - \beta}{R} \tag{5}$$

as shown at EQN (J1) on page S9-6.

EQNS (4) and (5) are exact.

GRAEME F. DENNES

DERIVATION OF THE DERIVATIVE OF THE WTE EXPRESSION

The WTE expression is shown at EQN J on page S9-6. The derivative of this expression with respect to radius R may be used to determine:

a. The true radius R_W at the central WTE peak.
b. The no-approximation or 'Perfect Löfgren A' solution, as shown on page S9-10.

Several of the authors of the papers provide expressions to calculate the radius, R_W at the maximum value of the central peak of the WTE function. However, because of mathematical simplifications used by the authors in the development of their solutions, the radius R_W so calculated is (only) a close approximation to the true radius R_W at the central peak.

To find the *true* radius R_W, equate the derivative to zero, and solve for the radius R (R_W), the radius at the true maximum value of the central WTE peak. The derivative is derived below, and is also shown at EQN R on page S9-9.

As there is no explicit solution for R for either the WTE function or its derivative, it is necessary to solve for R iteratively. Any iterative solution tool may be used to find the root of the derivative to the required precision (e.g. the Solver Tool in Microsoft Excel). The R so found is the true R_W.

The iterative solution tool may also be used to find the true absolute minimum WTE (the 'perfect Löfgren A' solution), for which the WTE at the three peaks are truly equal and truly minimised, to the accuracy specified. An example of some of the calculated results using this high accuracy technique is shown on page S9-10, and the complete results are shown on page S1-22.

The Derivative of the WTE Function

Starting with the WTE equation at EQN J on page S9-6.

$$WTE = \frac{\arcsin\left(\dfrac{R^2+L^2-M^2}{2LR}\right) - \beta}{R}$$

$$= \frac{1}{R}\arcsin\left(\frac{R^2+L^2-m^2}{2LR}\right) - \frac{\beta}{R}$$

Let y = WTE and solve for $\dfrac{dy}{dR}$

Starting with the Product Rule:

Let $uv = \dfrac{1}{R}\arcsin W$ where $W = \dfrac{R^2+L^2-M^2}{2LR}$

Then $\dfrac{d}{dR} uv = u \dfrac{dv}{dW} \cdot \dfrac{dW}{dR} + v \dfrac{du}{dR}$

So $$\frac{dy}{dR} = \frac{1}{R} \cdot \frac{1}{(1-W^2)^{\frac{1}{2}}} \cdot \frac{R^2 - L^2 + M^2}{2LR^2}$$

$$- \frac{1}{R^2} \cdot \arcsin W + \frac{\beta}{R^2}$$

So the derivative of the WTE expression is :

$$\frac{R^2 - L^2 + M^2}{2LR^3(1-U^2)^{\frac{1}{2}}} - \frac{\arcsin(U) - \beta}{R^2}$$

where $$U = \frac{R^2 + L^2 - M^2}{2LR}$$

for β in radians

NOTE : $R^2 - L^2 + M^2 = R^2 - 2Ld + d^2$

$R^2 + L^2 - M^2 = R^2 + 2Ld - d^2$

PROOF THAT THE PRODUCT OF THE NULL RADII IS EQUIVALENT TO LÖFGREN'S R_a^2 EQN (40)

The equations for the null radii have been derived at EQN (15) on page S12-7 and EQN (21) on page S12-10.
They are also shown at EQN C on page S9-3.

Starting with these equations:

$$R_{01} = \frac{2R_1R_2}{\left(1+\frac{1}{\sqrt{2}}\right)R_2 + \left(1-\frac{1}{\sqrt{2}}\right)R_1} \qquad (1)$$

and
$$R_{02} = \frac{2R_1R_2}{\left(1-\frac{1}{\sqrt{2}}\right)R_2 + \left(1+\frac{1}{\sqrt{2}}\right)R_1} \qquad (2)$$

Now, multiply (1) by (2):

$$= \frac{4(R_1R_2)^2}{\frac{1}{2}R_2^2 + \left(1\frac{1}{2}+\sqrt{2}\right)R_1R_2 + \left(1\frac{1}{2}-\sqrt{2}\right)R_1R_2 + \frac{1}{2}R_1^2} \qquad (3)$$

$$= \frac{4(R_1R_2)^2}{\left(1\frac{1}{2}+\sqrt{2}+1\frac{1}{2}-\sqrt{2}\right)R_1R_2 + \frac{1}{2}R_2^2 + \frac{1}{2}R_1^2}$$

$$= \frac{4(R_1R_2)^2}{3R_1R_2 + \frac{1}{2}R_2^2 + \frac{1}{2}R_1^2}$$

Multiply by 2:

$$= \frac{8(R_1R_2)^2}{6R_1R_2 + R_2^2 + R_1^2}$$

$$= \frac{8(R_1R_2)^2}{4R_1R_2 + (R_1+R_2)^2} \qquad (4)$$

= Löfgren's EQN (40) (EQN(4) page S2-3)

= a^2 term.

Therefore, Löfgren's a^2 term, shown at EQN(40) in his paper, is the product of the two null radii!

DERIVATION OF NULL RADII AVERAGE

From Stevenson's EQN (7) in Part 1:

$$\sin\theta_D = \frac{1}{2L}\left(x_o + \frac{x_m^2}{x_o}\right)$$

and $x_o' = \dfrac{x_m^2}{x_o}$

$$\text{so } \sin\theta_D = \frac{1}{2L}\left(x_o + x_o'\right)$$

$$\equiv \frac{R_{o1} + R_{o2}}{2L}$$

From Löfgren's EQN (41):

$$\cos\phi_o = p/R \equiv p/L$$

and as ϕ_o and β are complementary,

$$\sin\beta = p/L$$

$$\therefore \quad p = L\sin\beta$$

= Linear offset

$$= L \cdot \frac{R_{o1} + R_{o2}}{2L}$$

$$= \frac{R_{o1} + R_{o2}}{2}$$

= Null radii average

$\therefore$ Null radii average = Linear offset

DERIVATION OF OUTER RADIUS R_2 IN TERMS OF NULL RADII

From EQN (38) on page S6-11:

$$R_{02} = \frac{R_{01}R_2}{(2\sqrt{2}-2)R_{01} - (2\sqrt{2}-3)R_2} \qquad (1)$$

$$= \frac{R_{01}R_2}{2(\sqrt{2}-1)R_{01} - (2\sqrt{2}-3)R_2}$$

and $$\frac{1}{R_{02}} = \frac{2(\sqrt{2}-1)R_{01}}{R_{01}R_2} - \frac{(2\sqrt{2}-3)R_2}{R_{01}R_2}$$

$$= \frac{2(\sqrt{2}-1)}{R_2} - \frac{2\sqrt{2}-3}{R_{01}}$$

So $$\frac{2(\sqrt{2}-1)}{R_2} = \frac{2\sqrt{2}-3}{R_{01}} + \frac{1}{R_{02}}$$

ie $$\frac{2}{R_2} = \frac{2\sqrt{2}-3}{(\sqrt{2}-1)R_{01}} + \frac{1}{(\sqrt{2}-1)R_{02}} \qquad (2)$$

To simplify (2):

1. $$\frac{2\sqrt{2}-3}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{2.2 + 2\sqrt{2} - 3\sqrt{2} - 3}{2 + \sqrt{2} - \sqrt{2} - 1}$$

ie $$\frac{2\sqrt{2}-3}{\sqrt{2}-1} = 1 - \sqrt{2} \qquad (3)$$

2. $$\frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2+\sqrt{2}-\sqrt{2}-1}$$

ie $$\frac{1}{\sqrt{2}-1} = 1 + \sqrt{2} \qquad (4)$$

Now substite (3) and (4) into (2):

$$\frac{2}{R_2} = \frac{1-\sqrt{2}}{R_{01}} + \frac{1+\sqrt{2}}{R_{02}} \qquad (5)$$

$$= \frac{R_{01}(1+\sqrt{2}) + R_{02}(1-\sqrt{2})}{R_{01}R_{02}} \qquad (6)$$

So $$\frac{R_2}{2} = \frac{R_{01}R_{02}}{R_{01}(1+\sqrt{2}) + R_{02}(1-\sqrt{2})} \qquad (7)$$

and $$R_2 = \frac{2R_{01}R_{02}}{R_{01}(1+\sqrt{2}) + R_{02}(1-\sqrt{2})} \qquad (8)$$

DERIVATION OF INNER RADIUS R1 IN TERMS OF NULL RADII

From EQNS D1 and D3 on page S9-3:

$$\frac{2R_1R_2}{R_1+R_2} = \frac{2R_{01}R_{02}}{R_{01}+R_{02}} \quad (= R_W) \qquad (1)$$

ie $$2R_1R_2R_{01} + 2R_1R_2R_{02} = 2R_{01}R_{02}R_1 + 2R_{01}R_{02}R_2 \qquad (2)$$

ie $$R_1\left(R_2R_{01} + R_2R_{02} - R_{01}R_{02}\right) = R_{01}R_{02}R_2$$

ie $$R_1 = \frac{R_{01}R_{02}R_2}{R_{01}R_2 + R_{02}R_2 - R_{01}R_{02}} \qquad (3)$$

so $$\frac{1}{R_1} = \frac{R_{01}R_2}{R_{01}R_{02}R_2} + \frac{R_{02}R_2}{R_{01}R_{02}R_2} - \frac{R_{01}R_{02}}{R_{01}R_{02}R_2}$$

$$= \frac{1}{R_{02}} + \frac{1}{R_{01}} - \frac{1}{R_2} \qquad (4)$$

Substitute expression for R_2 derived on page S12-21:

ie $$\frac{1}{R_1} = \frac{1}{R_{02}} + \frac{1}{R_{01}} - \frac{R_{01}(1+\sqrt{2}) + R_{02}(1-\sqrt{2})}{2R_{01}R_{02}} \qquad (5)$$

$$= \frac{1}{R_{02}} + \frac{1}{R_{01}} - \left[\frac{1+\sqrt{2}}{2R_{02}} + \frac{1-\sqrt{2}}{2R_{01}}\right]$$

$$= \frac{1}{R_{02}} + \frac{1}{R_{01}} - \frac{1+\sqrt{2}}{2R_{02}} - \frac{1-\sqrt{2}}{2R_{01}} \qquad (6)$$

so $$\frac{2}{R_1} = \frac{2}{R_{02}} + \frac{2}{R_{01}} - \frac{1+\sqrt{2}}{R_{02}} - \frac{1-\sqrt{2}}{R_{01}} \qquad (7)$$

$$= \frac{2-1+\sqrt{2}}{R_{01}} + \frac{2-1-\sqrt{2}}{R_{02}}$$

ie $$\frac{2}{R_1} = \frac{1+\sqrt{2}}{R_{01}} + \frac{1-\sqrt{2}}{R_{02}} \quad (8)$$

$$= \frac{R_{01}(1-\sqrt{2}) + R_{02}(1+\sqrt{2})}{R_{01}R_{02}} \quad (9)$$

so $$\frac{R_1}{2} = \frac{R_{01}R_{02}}{R_{01}(1-\sqrt{2}) + R_{02}(1+\sqrt{2})} \quad (10)$$

so $$R_1 = \frac{2R_{01}R_{02}}{R_{01}(1-\sqrt{2}) + R_{02}(1+\sqrt{2})} \quad (11)$$

DERIVATION OF STEVENSON'S $R_W(x_p)$

FROM LÖFGREN'S $R_W(r^*)$

From Löfgren's EQN (36) : [EQNS (16) and (17), page S2-6]

$$r^* = R_W = \frac{L^2 - M^2}{L \cos\phi_0} \qquad \left(\equiv \frac{R^2 - D^2}{R \cos\phi_0} \right)$$

$$= \frac{L^2 - M^2}{L \sin\beta}$$

But $R_a^2 = L^2 - M^2$ Löfgren's EQN (39) [EQN (10) page S2-5]

ie $R_W = \dfrac{R_a^2}{L \sin\beta}$

But $\sin\beta = \dfrac{R_{01} + R_{02}}{2L}$ From Stevenson's EQN (7) [EQN (17) page S6-6]

ie
$$R_W = \frac{R_a^2}{L\left(\frac{R_{01} + R_{02}}{2L}\right)}$$

$$= \frac{2L \cdot R_a^2}{L(R_{01} + R_{02})}$$

$$= \frac{2 R_a^2}{R_{01} + R_{02}}$$

But $R_a^2 = R_{01} R_{02}$ From Löfgren's EQN (40) [EQN (4) page S2-3]

ie $R_W = \dfrac{2 R_{01} R_{02}}{R_{01} + R_{02}}$

$$\equiv \frac{2x_o x_o'}{x_o + x_o'}$$

But $x_o x_o' = x_m^2$

so $$R_W = \frac{2x_m^2}{x_o + x_o'}$$

Multiply by x_o:

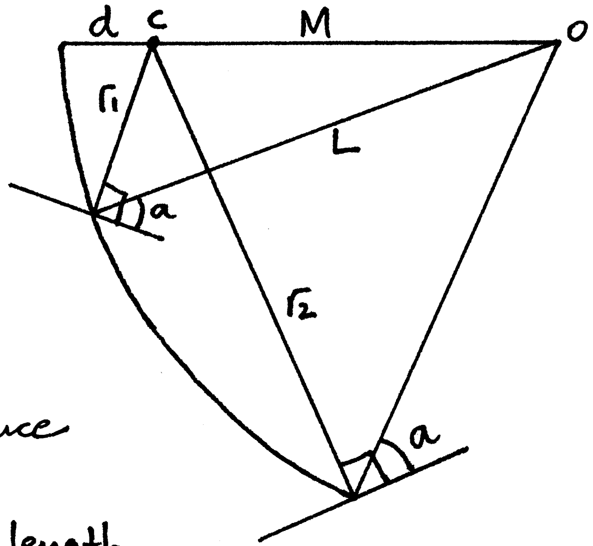
$$R_W = \frac{2x_o x_m^2}{x_o^2 + x_m^2}$$

so $$x_p = R_W = \frac{2x_o x_m^2}{x_o^2 + x_m^2}$$ (EQN (3), page S6-3)

per Stevenson's equation on page 218 of Part 1 of his article.

Thus, Stevenson's R_W (x_p) has been shown to be identical to Löfgren's R_W (r^*).

DERIVATION OF KESSLER/PISHA EQN (3).



d = overhang
M = Mounting Distance
= L − d
L = effective arm length
a = optimum offset angle = tracking angle at r_1, r_2
r_1 = inner null radius
r_2 = outer null radius
c = record centre
o = arm pivot point

From the diagram:

$$r_1^2 + L^2 - M^2 = 2 r_1 L \cos(90 - a) \qquad (11)$$
$$r_2^2 + L^2 - M^2 = 2 r_2 L \cos(90 - a) \qquad (12)$$

From (11):

$$\cos(90 - a) = \frac{r_1^2 + L^2 - M^2}{2 r_1 L} \qquad (13)$$

Substitute (13) into (12):

$$r_2^2 + L^2 - M^2 = 2 r_2 L \left[\frac{r_1^2 + L^2 - M^2}{2 r_1 L} \right]$$

$$= r_2\left[\frac{r_1^2 + L^2 - m^2}{r_1}\right]$$

ie, $$\frac{r_2^2 + L^2 - m^2}{r_2} = \frac{r_1^2 + L^2 - m^2}{r_1}$$

ie $$r_1(r_2^2 + L^2 - m^2) = r_2(r_1^2 + L^2 - m^2)$$

ie $$r_1 r_2^2 + r_1 L^2 - r_1 m^2 = r_2 r_1^2 + r_2 L^2 - r_2 m^2$$

ie $$r_2 m^2 - r_1 m^2 = r_2 r_1^2 + r_2 L^2 - r_1 r_2^2 - r_1 L^2$$

ie $$m^2(r_2 - r_1) = r_2 r_1^2 + r_2 L^2 - r_1 r_2^2 - r_1 L^2$$

ie $$m^2 = \frac{r_2(L^2 + r_1^2) - r_1(L^2 + r_2^2)}{r_2 - r_1}$$

$$\therefore\ M = \left[\frac{r_2(L^2 + r_1^2) - r_1(L^2 + r_2^2)}{r_2 - r_1}\right]^{1/2} \qquad (14)$$

EQN (14) is the same as Kessler/Pisha EQN (3).

EQN (14) reduces to $\left[L^2 - r_1 r_2\right]^{1/2}$ as shown in the analysis of their article.

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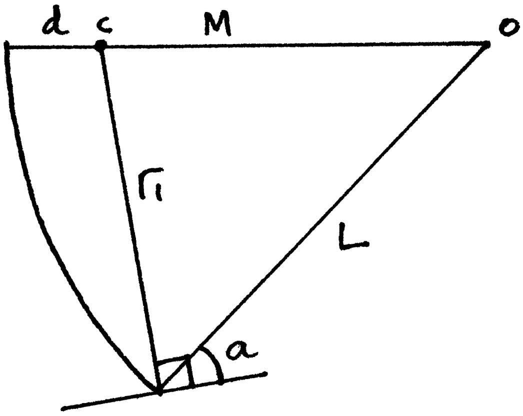
DERIVATION OF KESSLER/PISHA EQN (4)



From the diagram :

$$2 r_1 L \cos(90-a) = r_1^2 + L^2 - M^2$$

$$\therefore \quad M^2 = r_1^2 + L^2 - 2 r_1 L \cos(90-a)$$

$$\therefore \quad M = \left[r_1^2 + L^2 - 2 r_1 L \cos(90-a) \right]^{1/2} \qquad (15)$$

EQN (15) is the same as Kessler/Pisha EQN (4).

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THE EQUIVALENCE OF BAERWALD'S EQN (13) AND LÖFGREN'S EQN (2)

To show that Baerwald's EQN (13) is equivalent to Löfgren's EQN (2) and the standard tracking angle equation shown at EQN (F1) on page 59-4 of this analysis. Baerwald's notation will be used up until the end.

Baerwald's Notation

r = radius

r_m = mean groove radius

$= (r_1 r_2)^{\frac{1}{2}}$

$x = \dfrac{r}{L}$ $\qquad x_m = \dfrac{r_m}{L}$ $\qquad \delta = \dfrac{d}{L}$

η = tracking error $= \gamma - \alpha$

η' = WTE referred to mean groove radius

$= \eta \cdot \dfrac{r_m}{r}$

so $\eta = \eta' \cdot \dfrac{r}{r_m}$

and $\mathrm{WTE} = \dfrac{\eta}{r} = \dfrac{\eta'}{r_m}$

α = offset angle

γ = tracking angle

$-d$ = overhang (d = underhang)

From Baerwald's EQN (13):

$$\sin\gamma = \frac{x^2 - 2\delta - \delta^2}{2x} \quad \text{for } d < 0$$

$$= \frac{\left(\frac{r}{L}\right)^2 - 2\left(\frac{-d}{L}\right) - \left(\frac{-d}{L}\right)^2}{2r/L}$$

$$= \frac{\frac{r^2}{L^2} + \frac{2d}{L} - \frac{d^2}{L^2}}{2r/L}$$

$\times L^2$:

$$= \frac{r^2 + 2Ld - d^2}{2Lr}$$

$$\text{ie } \sin\gamma \equiv \frac{R^2 + 2Ld - d^2}{2LR} \qquad (1)$$

But $d = L - M$

$$\text{so } \sin\gamma = \frac{R^2 + 2L(L-M) - (L-M)^2}{2LR}$$

$$= \frac{R^2 + 2L^2 - 2LM - L^2 + 2LM - M^2}{2LR}$$

$$\text{ie } \sin\gamma = \frac{R^2 + L^2 - M^2}{2LR} \qquad (2)$$

Baerwald's EQN (13) is equivalent to the standard tracking angle equations at EQN (F) on page S9-4 of this analysis, and is equivalent to Lötgren's EQN (2).

THE EQUIVALENCE OF BAERWALD'S EQN (14) AND LÖFGREN'S EQN (33)

To show that Baerwald's EQN (14) is equivalent to Löfgren's EQN (33), shown at EQN (H1) on page S9-5 of this analysis. Löfgren's optimum equations for offset angle and overhang were derived directly from his EQN (33) for his 'Löfgren A' solution.

Baerwald's Notation:

r = radius

r_m = mean groove radius

$= (r_1 r_2)^{\frac{1}{2}}$

$x = \dfrac{r}{L}$ $\quad x_m = \dfrac{r_m}{L}$ $\quad \delta = \dfrac{d}{L}$

η = tracking error $= \gamma - \alpha$

η' = WTE referred to the mean groove radius

$= \eta \cdot \dfrac{r_m}{r}$

so $\eta = \eta' \cdot \dfrac{r}{r_m}$

and $\text{WTE} = \dfrac{\eta}{r} = \dfrac{\eta'}{r_m}$

α = offset angle

$-d$ = overhang

γ = tracking angle

From Baerwald's EQN (14):

$$\eta' = \frac{x_m}{2\cos\alpha}\left[1 - \frac{2\delta + \delta^2}{x^2} - \frac{2\sin\alpha}{x}\right]$$

then $WTE = \frac{x_m}{2r_m \cos\alpha}\left[\quad \text{As above} \quad \right]$

$$= \frac{r_m}{L} \cdot \frac{1}{2r_m \cos\alpha}\left[\quad '' \quad \right]$$

$$= \frac{1}{2L\cos\alpha}\left[\quad '' \quad \right]$$

$$= \frac{1}{2L\cos\alpha}\left[\frac{x^2 - 2\delta - \delta^2 - 2x\sin\alpha}{x^2}\right]$$

But $\delta = \frac{d}{L}$ where $d < 0$

so $WTE = \frac{1}{2L\cos\alpha} \dfrac{\left[\left(\frac{r}{L}\right)^2 - 2\left(\frac{-d}{L}\right) - \left(\frac{-d}{L}\right)^2 - \frac{2r}{L}\sin\alpha\right]}{r^2/L^2}$

$$= \frac{1}{2L\cos\alpha} \dfrac{\left[\frac{r^2}{L^2} + \frac{2d}{L} - \frac{d^2}{L^2} - \frac{2r\sin\alpha}{L}\right]}{r^2/L^2}$$

$$= \frac{L^2}{2Lr^2\cos\alpha}\left[\frac{r^2}{L^2} + \frac{2d}{L} - \frac{d^2}{L^2} - \frac{2r\sin\alpha}{L}\right]$$

$$= \frac{L}{2r^2\cos\alpha}\left[\frac{r^2}{L^2} + \frac{2Ld}{L^2} - \frac{d^2}{L^2} - \frac{2Lr\sin\alpha}{L^2}\right]$$

$$= \frac{L}{2L^2r^2\cos\alpha}\left[r^2 + 2Ld - d^2 - 2Lr\sin\alpha\right]$$

ie $WTE = \dfrac{1}{2Lr^2\cos\alpha}\left[r^2 + 2Ld - d^2 - 2Lr\sin\alpha\right]$

But $d = L - M$

so $WTE = \dfrac{1}{2Lr^2\cos\alpha}\left[r^2 + 2L(L-M) - (L-M)^2 - 2Lr\sin\alpha\right]$

$$= \frac{1}{2Lr^2\cos\alpha}\left[r^2 + 2L^2 - 2LM - L^2 + 2LM - M^2 - 2Lr\sin\alpha\right]$$

$$= \frac{1}{2Lr^2\cos\alpha}\left[r^2 + L^2 - M^2 - 2Lr\sin\alpha\right]$$

$\div\ 2LR$

so $WTE = \dfrac{1}{r\cos\alpha}\left[\dfrac{r^2 + L^2 - M^2}{2Lr} - \sin\alpha\right]$

$$\equiv \frac{1}{R\cos\beta}\left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]$$

Baerwald's EQN (14) is equivalent to Löfgren's EQN (33), shown above.

THE EQUIVALENCE OF (PART OF) BAERWALD'S EQN (16b) AND LÖFGREN'S Ra^2 EQN (40)

From Baerwald's EQN (16b), as shown at EQN (12) on page S3-4 of this analysis, the right hand term inside the square brackets is :

$$\frac{2\,Xm^2}{\rho^2+1}$$

From EQN (14) on page S3-5 of this analysis, this term has been expanded to:

$$\frac{2R_1^2R_2^2}{\frac{1}{4}(R_1+R_2)^2+R_1R_2} = \frac{8R_1^2R_2^2}{(R_1+R_2)^2+4R_1R_2}$$

which we may observe is identical to Lötgren's Ra^2 equation at his EQN (40).

Therefore, Baerwald's expression for Ra^2 is identical to Lötgren's.

EQUIVALENCE OF BAERWALD'S EQN (18) AND LÖFGREN'S EQN (36) FOR RW

Using Baerwald's notation:

From Baerwald's EQN (18):

$$x_0 = \frac{-2\delta - \delta^2}{\sin\alpha}$$

ie
$$\frac{r_0}{L} = \frac{-2\left(\frac{-d}{L}\right) - \left(\frac{-d}{L}\right)^2}{\sin\alpha}$$

$$= \frac{\frac{2d}{L} - \frac{d^2}{L^2}}{\sin\alpha}$$

so
$$r_0 = \frac{2d - \frac{d^2}{L}}{\sin\alpha}$$

$$= \frac{2Ld - d^2}{L\sin\alpha}$$

ie
$$Rw = \frac{2Ld - d^2}{L\sin\beta}$$

But $d = L - M$

so
$$Rw = \frac{2L(L-M) - (L-M)^2}{L\sin\beta}$$

$$= \frac{2L^2 - 2LM - L^2 + 2LM - M^2}{L\sin\beta}$$

ie $\quad R_W = \dfrac{L^2 - M^2}{L \sin\beta}$

This is equivalent to Lötgren's EQN (36)

THE EQUIVALENCE OF BAERWALD'S EQN(10) AND LÖFGREN'S EQN(22)

From Baerwald's EQN (10):

$$\varepsilon = \frac{v\eta}{r\Omega} \equiv \frac{V}{\Omega} \cdot \text{WTE}$$

From Löfgren's EQN (22):

$$\varepsilon = \frac{V}{\Omega} \cdot \frac{\delta}{R} \equiv \frac{V}{\Omega} \cdot \text{WTE}$$

Therefore, Baerwald's EQN (10) is identical to Löfgren's EQN (22).

THE EQUIVALENCE OF BAERWALD'S r_0 AT EQN (16e) AND LÖFGREN'S r^* AT EQN (37)

From Baerwald's EQN (16e):

$$r_0 = \frac{2 r_1 r_2}{r_1 + r_2} \equiv \frac{2 R_1 R_2}{R_1 + R_2} = R_W$$

From Löfgren's EQN (37):

$$r^* = \frac{2 r_1 r_2}{r_1 + r_2} \equiv \frac{2 R_1 R_2}{R_1 + R_2} = R_W$$

Therefore, Baerwald's EQN (16e) for R_W is identical to Löfgren's EQN (37) for R_W.

DERIVATION OF SEAGRAVE'S EQN(18)
WTE AT RADIUS R

From the tracking angle equation:

$$\sin\alpha = \frac{R^2 + 2Ld - d^2}{2LR} \qquad (1)$$

$$= \frac{R}{2L} + \frac{d}{R} - \frac{d^2}{2LR}$$

$$= \frac{R}{2L} + \frac{d}{R}\left(1 - \frac{d}{2L}\right) \qquad (2)$$

EQN (2) is identical to Seagrave's EQN (12).

The tracking error is given by:

$$\phi = \alpha - \beta \qquad (3)$$

but EQN (3) does not lend itself to further analysis because the arcsin function is involved.

Seagrave used an approximation to allow further analysis to be done, it is based on the Sine Rule:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Let $A = \alpha - \beta$, and $B = \beta$

then

$$\sin(\alpha - \beta + \beta) = \sin(\alpha-\beta)\cos\beta + \cos(\alpha-\beta)\sin\beta$$

ie $\sin\alpha = \sin(\alpha-\beta)\cos\beta + \cos(\alpha-\beta)\sin\beta \qquad (5)$

EQN (5) is mathematically correct in every sense.

We now introduce two approximations:

1. For $(\alpha-\beta)$ small, $\sin(\alpha-\beta) \approx (\alpha-\beta)$ radians.

2. For $(\alpha-\beta)$ small, $\cos(\alpha-\beta) \approx 1$

So from (5):

$$\sin\alpha \approx (\alpha-\beta)\cos\beta + 1.\sin\beta$$

ie $$\sin\alpha \approx (\alpha-\beta)\cos\beta + \sin\beta \qquad (6)$$

EQN (6) is identical to Seagrave's EQN (13).

From (6):

$$\alpha-\beta \approx \frac{\sin\alpha - \sin\beta}{\cos\beta} \text{ radians} \qquad (7)$$

EQN (7) is identical to Seagrave's EQN (14).

From EQN (2), Seagrave introduced:

$$\text{Let } D_1 = d\left(1-\frac{d}{2L}\right) \qquad (8)$$

Note that this is only a shorthand, not an approximation. He used it to simplify the notation.

So from EQN (2) above:

$$\sin\alpha = \frac{R}{2L} + \frac{D_1}{R} \qquad (9)$$

From EQNs (7) and (9):

$$\alpha - \beta \approx \frac{\sin\alpha - \sin\beta}{\cos\beta}$$

$$\approx \frac{\frac{R}{2L} + \frac{D_1}{R} - \sin\beta}{\cos\beta} \qquad (10)$$

From (10) we can express the WTE as follows:

$$WTE = \frac{\alpha - \beta}{R}$$

$$= \frac{\frac{R}{2L} + \frac{D_1}{R} - \sin\beta}{R\cos\beta}$$

$$WTE = \left[\frac{1}{2L} + \frac{D_1}{R^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta} \qquad (11)$$

radians per unit length

where $D_1 = d\left(1 - \frac{d}{2L}\right)$ as before

EQN (11) is identical to Seagrove's EQN (18)

THE EQUIVALENCE OF SEAGRAVE'S EQN(18) AND LÖFGREN'S EQN(33)

These equations calculate the WTE at radius R.

From Seagrave's EQN (18):

$$WTE = \left[\frac{1}{2L} + \frac{D_1}{R^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta} \qquad (1)$$

radians per unit length

where $D_1 = d\left(1-\frac{d}{2L}\right)$ per Seagrave's (15)

Substituting for D_1:

$$WTE = \left[\frac{1}{2L} + \frac{d\left(1-\frac{d}{2L}\right)}{R^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{R^2}{2LR} + \frac{2Ld\left(1-\frac{d}{2L}\right)}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta}$$

$$= \left[\frac{R^2 + 2Ld - d^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (2)$$

As $2Ld - d^2 = L^2 - M^2$, then

$$WTE = \left[\frac{R^2 + L^2 - M^2}{2LR} - \sin\beta\right]\frac{1}{R\cos\beta} \qquad (3)$$

EQN (3) is identical to Löfgren's EQN (33), so Seagrave's EQN (18) is identical to Löfgren's EQN (33).

EQNS (2) and (3) are equivalent.

DERIVATION OF SEAGRAVE'S EQN (19), WTE AT RW

From Seagrave's EQN (18):

$$WTE = \left[\frac{1}{2L} + \frac{D_1}{R^2} - \frac{\sin\beta}{R}\right]\frac{1}{\cos\beta}$$ radians per unit length

$$\left[\text{where } D_1 = d\left(1 - \frac{d}{2L}\right)\right]$$

so $$WTE = \frac{1}{2L\cos\beta} + \frac{D_1}{R^2\cos\beta} - \frac{\sin\beta}{R\cos\beta} \qquad (1)$$

The radius RW occurs at the minimum of the WTE function, so differentiate with respect to radius R, then solve for the root R of the derivative to obtain RW, as follows:

$$\frac{d}{dR} = 0 - \frac{2D_1}{R^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta}$$

being that L, β and d are constants.

$$= -\frac{2D_1}{R^3\cos\beta} + \frac{\sin\beta}{R^2\cos\beta} \qquad (2)$$

Equate to zero and solve for R:

ie $$\frac{\sin\beta}{R^2\cos\beta} - \frac{2D_1}{R^3\cos\beta} = 0 \qquad (3)$$

$\times R^2$:

$$\frac{\sin\beta}{\cos\beta} - \frac{2D_1}{R\cos\beta} = 0$$

ie $\quad R\cos\beta \, . \, \sin\beta = 2D_1\cos\beta$

so $\quad R = \dfrac{2D_1\cos\beta}{\cos\beta\sin\beta}$

ie $\quad R_W = \dfrac{2D_1}{\sin\beta} \qquad (4)$

Now substitute R_W for R in Seagrave's EQN (18):

So WTE at $R = R_W$ is:

$$\left[\frac{1}{2L} + \frac{D_1}{R_W^2} - \frac{\sin\beta}{R_W}\right]\frac{1}{\cos\beta} \quad \text{radians per unit length}$$

$$= \left[\frac{1}{2L} + \frac{D_1\sin^2\beta}{(2D_1)^2} - \frac{\sin\beta.\sin\beta}{2D_1}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{1}{2L} + \frac{\sin^2\beta}{4D_1} - \frac{\sin^2\beta}{2D_1}\right]\frac{1}{\cos\beta}$$

$$= \left[\frac{1}{2L} + \frac{\sin^2\beta}{D_1}\left(\frac{1}{4} - \frac{1}{2}\right)\right]\frac{1}{\cos\beta}$$

$$\text{So WTE at } R_W = \left[\frac{1}{2L} - \frac{\sin^2\beta}{4D_1}\right]\frac{1}{\cos\beta} \qquad (5)$$

radians per unit length

The above equation is identical to Seagrave's EQN (19).

DERIVATION OF BAUER'S TRACKING ANGLE ϕ EQN (5)

From Bauer's EQN (2) : (Let α = Bauer's ϕ)

$$R^2 + L^2 - 2RL\sin\alpha = M^2$$

ie

$$\sin\alpha = \frac{R^2 + L^2 - M^2}{2LR}$$

$$= \frac{R}{2L} + \frac{L^2 - M^2}{2LR}$$

As $M = L - d$, then

$$\sin\alpha = \frac{R}{2L} + \frac{L^2 - (L-d)^2}{2LR}$$

$$= \frac{R}{2L} + \frac{L^2 - (L^2 - 2Ld + d^2)}{2LR}$$

$$= \frac{R}{2L} + \frac{2Ld - d^2}{2LR}$$

Bauer's Simplifying Approximations

1. As $d^2 << 2Ld$, then ignore d^2, so

$$\sin\alpha = \frac{R}{2L} + \frac{d}{R}$$

2. Let $\sin\alpha = \alpha$ radians, so

$$\alpha = \frac{R}{2L} + \frac{d}{R} \text{ radians}$$

The equation above is identical to Bauer's EQN (5)

THE EQUIVALENCE OF BAUER'S (1949) EQN (2) AND SEAGRAVE'S EQN (26)

From Bauer's EQN (2) in his 1949 paper:

$$\sin\beta_i = \frac{1}{L\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - \frac{R_1}{2}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right]}$$

$$= \frac{1}{L\left[\frac{R_1+R_2}{R_1R_2} - \frac{1}{2R_1} - \frac{R_1}{2R_2^2}\right]}$$

$$= \frac{1}{\frac{L}{R_1}\left[\frac{R_1+R_2}{R_2} - \frac{1}{2} - \frac{R_1^2}{2R_2^2}\right]}$$

$$= \frac{R_1}{L\left[\frac{R_1}{R_2} + 1 - \frac{1}{2} - \frac{R_1^2}{2R_2^2}\right]}$$

$$= \frac{R_1}{L\left[1 - \frac{1}{2}\left(1 - \frac{2R_1}{R_2} + \frac{R_1^2}{R_2^2}\right)\right]}$$

$$= \frac{R_1}{L\left[1 - \frac{1}{2}\left(1 - \frac{R_1}{R_2}\right)^2\right]}$$

The equation above is identical to Seagrave's EQN (26).

The "Best" Non-Optimum 'Löfgren A' Designs

On pages S9-11 and S9-13 are procedures and equations for calculating the "best" possible 'Löfgren A' alignment for those situations where either the offset angle or the overhang is non-optimum. This can arise in practice if one of these is either fixed at a non-optimum setting, of has an insufficient adjustment range which prevents the optimum setting being achieved. (The reader is referred to the Section 9 pages before continuing).

For these non-optimum situations, the resulting WTE is not as small as that provided by the standard 'Löfgren A' alignment, but nevertheless is as small as possible for the circumstances. The outcome will be better than otherwise.

In principle, the non-optimum designs are based on equalising the WTE at *two* of the three key radii, R_1, R_W, and R_2.

Detailed discussion of non-optimum alignments is presented in the papers of Baerwald, Bauer (1949) and Seagrave. It turns out that the "best" compromise designs provided by these authors are *identical*.

The "best" non-optimum procedures and equations in Section 9 are drawn from the equations of Bauer (1949), and improved by Seagrave. (Bauer's offset angle and overhang equations are approximations. Seagrave discusses this.)

The equations given in Section 9 as solutions for the non-optimum designs will now be derived. The equations for calculating the "best" overhang for a non-optimum offset angle will be derived first, followed by the derivation of the equations for calculating the "best" offset angle for a non-optimum overhang.

Seagrave's Equations for WTE at R_1, R_W and R_2

Seagrave's EQN (18) gives the WTE at radius R, and we make use of this equation for calculating the WTE at radius R_1 and R_2. Seagrave's EQN (19) gives the WTE at radius R_W. So from these equations:

$$\text{WTE at } R_1 = \left[\frac{1}{2L} + \frac{D}{R_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} \qquad (1)$$

$$\text{WTE at } R_2 = \left[\frac{1}{2L} + \frac{D}{R_2^2} - \frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} \qquad (2)$$

$$\text{WTE at } R_W = \left[\frac{1}{2L} - \frac{\sin^2\beta}{4D}\right]\frac{1}{\cos\beta} \qquad (3)$$

$$\text{where } D = \frac{2Ld - d^2}{2L} \qquad (4)$$

Seagrave's Shorthand Notation

As noted in the analysis of Seagrave's paper in Section 5, Seagrave uses a shorthand in the development of his equations as follows:

$$\text{Let } D = \frac{2Ld - d^2}{2L} \tag{5}$$

where d = overhang,

and, conversely,

$$d = L - \left[L^2 - 2LD\right]^{\frac{1}{2}} \tag{6}$$

This shorthand is also used in the non-optimum 'Löfgren A' equations presented at pages S9-11 and S9-13, and will also be used here in the derivation of the non-optimum equations.

Calculation of "Best" Overhang For Non-Optimum Offset Angle

In the procedure presented at page S9-11 in Section 9, two offset angles are firstly calculated. The first angle is the standard (ideal) 'Löfgren A' offset angle β_0 for the given arm length and groove radii, and the second angle is a "special" offset angle β_1 calculated by EQN (1) on page S9-11. Depending on the value of the non-optimum offset angle β of the tonearm, the "best" overhang is calculated using one of the three equations EQN (2) on page S9-11, EQN (3) or EQN (4) on page S9-12, as described therein. These three equations will now be derived.

Case 1: $\beta \geq \beta_0$

In this case, we set WTE R_1 = -WTE R_W and solve for D as follows:

Let $$\left[\frac{1}{2L}+\frac{D}{R_1^2}-\frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L}-\frac{\sin^2\beta}{4D}\right]\frac{1}{\cos\beta} \qquad (7)$$

from which we obtain:

$$D^2 + D\left(\frac{R_1^2}{L} - R_1\sin\beta\right) - \frac{\sin^2\beta\, R_1^2}{4} = 0 \qquad (8)$$

Solving the above quadratic for D:

$$D = \frac{R_1}{2}\left(\sin\beta - \frac{R_1}{L}\right) \pm \frac{R_1}{2}\left[\left(\sin\beta - \frac{R_1}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} \qquad (9)$$

$$= \frac{R_1}{2}\left[\left[\left(\sin\beta - \frac{R_1}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} + \sin\beta - \frac{R_1}{L}\right] \qquad (10)$$

EQN (10) above is the same as EQN (2) on page S9-11.

It is also the same as Seagrave's EQN (27a) and Bauer's (1949) EQN (5).

Case 2: $\beta_1 < \beta < \beta_0$

In this case, we set WTE R_2 = -WTE R_W and solve for D as follows:

$$\text{Let} \left[\frac{1}{2L} + \frac{D}{R_2^2} - \frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L} - \frac{\sin^2\beta}{4D}\right]\frac{1}{\cos\beta} \qquad (11)$$

from which we obtain:

$$D^2 + D\left(\frac{R_2^2}{L} - R_2\sin\beta\right) - \frac{\sin^2\beta\, R_2^2}{4} = 0 \qquad (12)$$

Solving the above quadratic for D:

$$D = \frac{R_2}{2}\left(\sin\beta - \frac{R_2}{L}\right) \pm \frac{R_2}{2}\left[\left(\sin\beta - \frac{R_2}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} \qquad (13)$$

$$= \frac{R_2}{2}\left[\left[\left(\sin\beta - \frac{R_2}{L}\right)^2 + \sin^2\beta\right]^{\frac{1}{2}} + \sin\beta - \frac{R_2}{L}\right] \qquad (14)$$

EQN (14) above is the same as EQN (3) on page S9-12.

It is also the same as Seagrave's EQN (27b) and Bauer's (1949) EQN (4).

Case 3: $\beta \leq \beta_1$

In this case, we set WTE R_2 = -WTE R_1 and solve for D as follows:

Let $$\left[\frac{1}{2L} + \frac{D}{R_2^2} - \frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L} + \frac{D}{R_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} \qquad (15)$$

from which we obtain:

$$\frac{D}{R_1^2} + \frac{D}{R_2^2} = \frac{\sin\beta}{R_1} + \frac{\sin\beta}{R_2} - \frac{1}{L} \qquad (16)$$

Solving for D, we obtain:

$$D = \frac{\sin\beta\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{1}{L}}{\frac{1}{R_1^2} + \frac{1}{R_2^2}} \qquad (17)$$

EQN (17) above is the same as EQN (4) on page S9-12.

It is also the same as Seagrave's EQN (27c) and Bauer's (1949) EQN (3).

Calculation of "Best" Offset Angle For Non-Optimum Overhang

In the procedure presented at page S9-13 in Section 9, two overhang values are firstly calculated. The first overhang value is D_0 and is the standard (ideal) 'Löfgren A' overhang for the given arm length and groove radii. The second overhang value is a "special" overhang D_1 calculated by EQN (1) on page S9-13. Depending on the value of the non-optimum overhang D of the tonearm, the "best" offset angle is calculated using one of the three equations EQN (2), EQN (3) or EQN (4) on page S9-14, as described therein. These three equations will now be derived.

Case 1: $D \geq D_0$

In this case, we set WTE R_1 = -WTE R_W and solve for β as follows:

Let
$$\left[\frac{1}{2L} + \frac{D}{R_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L} - \frac{\sin^2\beta}{4D}\right]\frac{1}{\cos\beta} \qquad (18)$$

from which we obtain:

$$\sin^2\beta + \frac{4D}{R_1}\sin\beta - \frac{4D^2}{R_1^2} - \frac{4D}{L} = 0 \qquad (19)$$

Solving the above quadratic for $\sin\beta$:

$$\sin\beta = 2\left[\frac{2D^2}{R_1^2} + \frac{D}{L}\right]^{\frac{1}{2}} - \frac{2D}{R_1} \qquad (20)$$

EQN (20) above is the same as EQN (2) on page S9-14.

It is also the same as Bauer's (1949) EQN (10), apart from the accuracy improvement suggested by Seagrave.

Case 2: $D_1 < D < D_0$

In this case, we set WTE R_2 = -WTE R_W and solve for β as follows:

$$\text{Let}\left[\frac{1}{2L}+\frac{D}{R_2^2}-\frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L}-\frac{\sin^2\beta}{4D}\right]\frac{1}{\cos\beta} \qquad (21)$$

From which we obtain :

$$\sin^2\beta + \frac{4D.\sin\beta}{R_2} - \frac{4D^2}{R_2^2} - \frac{4D}{L} = 0 \qquad (22)$$

Solving the above quadratic for $\sin\beta$:

$$\sin\beta = 2\left[\frac{2D^2}{R_2^2}+\frac{D}{L}\right]^{\frac{1}{2}} - \frac{2D}{R_2} \qquad (23)$$

EQN (23) above is the same as EQN (3) on page S9-14.

It is also the same as Bauer's (1949) EQN (9).

Case 3: $D \leq D_1$

In this case, we set WTE R_2 = -WTE R_1 and solve for β as follows:

$$\text{Let}\left[\frac{1}{2L} + \frac{D}{R_2^2} - \frac{\sin\beta}{R_2}\right]\frac{1}{\cos\beta} = -\left[\frac{1}{2L} + \frac{D}{R_1^2} - \frac{\sin\beta}{R_1}\right]\frac{1}{\cos\beta} \qquad (24)$$

from which we obtain:

$$\frac{\sin\beta}{R_1} + \frac{\sin\beta}{R_2} = \frac{D}{R_1^2} + \frac{D}{R_2^2} + \frac{1}{L} \qquad (25)$$

ie

$$\sin\beta = \frac{D\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{1}{L}}{\frac{1}{R_1} + \frac{1}{R_2}} \qquad (26)$$

EQN (26) above is the same as EQN (4) on page S9-14.

It is also the same as Bauer's (1949) EQN (8).

Löfgren 'A' and 'B' Zero Radii Calculations

Page S12-11 shows the derivation of the true zero radii (roots) of the standard tracking error equation using the standard quadratic (second-order) solution. *The formula applies to all alignments.*

This results in: $\mathrm{R}_{01/02} = p \mp \mathrm{SQRT}\,(p^2 - \mathrm{Ra}^2)$ (1)

where $p = \mathrm{L}.\sin(\beta)$ (2)

However, the Ra^2 term itself is alignment-dependent.

'Löfgren A' Zero Radii:
The **zero radii** are shown on page S9-8 with Ra^2 shown on page S9-4. Ra^2 (as a^2) is shown in Löfgren's paper at EQN (40). Ra^2 is proven to be equal to ($2\mathrm{Ld} - \mathrm{d}^2$) on page S10-28 and equal to ($\mathrm{L}^2 - \mathrm{M}^2$) on page S10-30. Note: this equivalence is independent of the type of alignment. It applies to *all alignments.*

The $\mathbf{Ra^2}$ term is derived on pages S10-28 to S10-30 and is shown on page S10-30 at EQN (62) as follows:

$$\mathrm{Ra}^2 = \frac{8\,\mathrm{R}_1^2\,\mathrm{R}_2^2}{(\mathrm{R}_1 + \mathrm{R}_2)^2 + 4\,\mathrm{R}_1\,\mathrm{R}_2} \qquad (3)$$

For the **zero radii**, substitute (2) and (3) into (1) to obtain the 'Löfgren A' zero radii:

$$\mathrm{R}_{01/02} = \mathrm{L}\sin(\beta) \mp \sqrt{\mathrm{L}^2\sin^2(\beta) - \frac{8\,\mathrm{R}_1^2\,\mathrm{R}_2^2}{(\mathrm{R}_1 + \mathrm{R}_2)^2 + 4\,\mathrm{R}_1\,\mathrm{R}_2}} \qquad (4)$$

'Löfgren B' Zero Radii:
The *zero radii* are shown on page S9-19 with Ra^2 shown on page S9-18. Ra^2 (as a^2) is shown in Löfgren's paper at EQN (48). Ra^2 is proven to be equal to ($2\mathrm{Ld} - \mathrm{d}^2$) on page S10–28 and equal to ($\mathrm{L}^2 - \mathrm{M}^2$) on page S10–30. To repeat, this equivalence is independent of alignment type.

The $\mathbf{Ra^2}$ term is derived on pages S11-7 to S11-10 and is shown on page S11-10 at EQN (18) as follows:

$$\mathrm{Ra}^2 = \frac{3\mathrm{R}_1\mathrm{R}_2\,[\,p\,(\mathrm{R}_1 + \mathrm{R}_2) - \mathrm{R}_1\mathrm{R}_2\,]}{\mathrm{R}_1^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2^2} \qquad (5)$$

For the **zero radii**, substitute (2) and (5) into (1) to obtain the 'Löfgren B' zero radii:

$$\mathrm{R}_{01/02} = \mathrm{L}\sin(\beta) \mp \sqrt{\mathrm{L}^2\sin^2(\beta) - \frac{3\mathrm{R}_1\mathrm{R}_2\,[\,\mathrm{L}\sin(\beta)\,(\mathrm{R}_1 + \mathrm{R}_2) - \mathrm{R}_1\mathrm{R}_2\,]}{\mathrm{R}_1^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2^2}} \qquad (6)$$

Note: The above zero radii formulas are not based on previously unknown formulas or techniques. They come from applying Löfgren's EQNs (40) and (48) for Ra^2 with the standard mathematical second-order polynomial solution for the tracking error equation.