The Spiral of Archimedes, the Logarithmic / Exponential Spiral and the Hyperbolic Spiral, and their Arc Lengths and Tangent Angles

By Patrick Bruskiewich

Abstract

This short paper looks at three Spirals – the Spiral of Archimedes, the Logarithmic / Exponential Spiral and the Hyperbolic Spiral – and their arc lengths, and tangent angles. This manuscript was written in 1982 but remained lost in the author's library until recently rediscovered.

1.0 The Spiral of Archimedes

One very nice problem in calculus is determining the length of paper tightly wrapped up in a roll of paper. Such a system is known as the Spiral of Archimedes, or a uniformly wrapped system.

The Spiral is similar to that of a Nautilus Shell, but is governed by the constraint that the distance between each layer of the roll is a constant spacing c. Other Spirals, like the Logarithmic / Exponential or the Hyperbolic Spiral, have no such spacing constraint.

Consider a long piece of paper wrapped into a roll with a starting radius a, and an ending radius b, and a spacing between layers of c. Let the angle θ, expressed in radians, be the angle from the origin to the point at radius r.

On a paper wrapped in a roll, by inspection, we find that if c is the spacing between layers that

$$d\theta / 2\pi = dr / c$$

This means that

$$d\theta / dr = (2\pi / c)$$

which defines the infinitesimal in r such that

$$dr = (c / 2\pi)\, d\theta$$

which has the solution

$$r(\theta) = (c / 2\pi)\, \theta + r_0$$

where θ is in radians and r_0 is the starting radius.

Consider now the arc length along the paper. The increment of arc length ds is given by

$$ds^2 = (r d\theta)^2 + (dr)^2$$

or equivalently

$$ds = \sqrt{[\, 1 + r^2 (d\theta/dr)^2 \,]}\ \ dr$$

Let

$$d\theta / dr = (2\pi / c)$$
$$= (1/k)$$

From which we get

$$ds = (1/k) \sqrt{[\, k^2 + r^2 \,]}\ \ dr$$

Now take the integral

$$\int ds = (1/k) \int \sqrt{[\, k^2 + r^2 \,]}\ \ dr$$

which means that the arc length for the uniformly wrapping spiral is

$$s = (1/2k) [\, r\sqrt{(r^2 + k^2)} + k^2 \ln (r + \sqrt{(r^2 + k^2)}) \,]$$

Now consider an ψ angle that is perpendicular to the radius line which is at angle θ to the horizontal, such that

$$\alpha = \theta + \psi$$

where α is the tangent angle of the point measured with regards to the horizontal.

By inspection we see that

$$\tan \psi = r\, d\theta / dr$$
$$= r / (dr / d\theta)$$

so that

$$\psi = \tan^{-1} (r\, d\theta / dr)$$

For the Spiral of Archimedes

$$\psi = \tan^{-1} (\theta)$$

so that the tangent angle α is

$$\alpha = \theta + \tan^{-1} (\theta)$$

2.0 The Logarithmic or Exponential Spiral

Define a Logarithmic or Exponential Spiral with radius has being given by

$$r(\theta) = a \exp (b\theta)$$

The spacing of the layers in this spiral

$$r(\theta + 2\pi) - r(\theta) = a (\exp (b(\theta + 2\pi)) - \exp (b\theta))$$
$$= a \exp (b\theta) [\exp (2\pi b) - 1]$$

$\ldots = r(\theta)\ [\exp(2\pi b) - 1]$

Note then that the spacing is a function of the radius. A more straightforward angular measure for this spiral is

$$d\theta / dr = (\exp(-b\theta)/ ab)$$

The increment of arc for this type of spiral is

$$ds = \sqrt{[\ 1 + r^2 (\exp(-b\theta)/ ab)^2\ ]}\ dr$$
$$= \sqrt{[\ (ab)^2 + (a \exp(b\theta))^2 (\exp(-2b\theta))\ ]} \exp(b\theta)\ d\theta$$
$$= a \sqrt{[\ b^2 + (\exp(2b\theta) (\exp(-2b\theta))\ ]} \exp(b\theta)\ d\theta$$
$$= a \sqrt{[\ b^2 + 1\ ]} \exp(b\theta)\ d\theta$$

By straightforward integration we find the arc length is

$$s(\theta) = (a / b) [\sqrt{(1 + b^2)}\ ] \exp(b\theta)$$

Consider next

$$\psi = \tan^{-1} (r\, d\theta / dr)$$

For this type of spiral

$$\psi = \tan^{-1} (r\, d\theta / dr)$$

$$= \tan^{-1} (1/ b)$$

which you note is a constant.

The tangent angle α for the Logarithmic or Exponential Spiral is

$$\alpha = \theta + \tan^{-1}(1/b)$$

This Spiral is sometimes called the *Equiangular Spiral* because of the constancy in ψ

The fact that many growing systems change their size in proportion to their size at any particular moment means that Logarithmic Spiral are an integral part of nature.

3.0 The Hyperbolic Spiral

The *Hyperbolic Spiral* is defined as

$$r\theta = a$$

where a is a positive constant.

This curve is called a Hyperbolic Spiral because of the resemblance of the equation to $xy = a$ which represents a hyperbola in Cartesian coordinates.

$$d(r\theta) = dr\,\theta + r\,d\theta = 0$$

from which we get

$$d\theta / dr = -\theta / r$$
$$= -a / r^2$$

The arc length is

$$ds = \sqrt{[1 + r^2 (d\theta/dr)^2]} \; dr$$

which for the Hyperbolic Spiral is

$$ds = (1/r) \sqrt{[r^2 + a^2]} \; dr$$

the integral of which is

$$s = \int (1/r) \sqrt{[r^2 + a^2]} \; dr$$
$$= \sqrt{[r^2 + a^2]} - a \ln [(a / r) + (1/r) \sqrt{[r^2 + a^2]}]$$

Consider the asymptotes for this curve.

At very small angle θ, r is very large which means that the point on the hyperbolic spiral arm is far to the right with y = a.

At a very large angle, the radius approaches zero, and so the Hyperbolic Spiral is spiralling inwards towards the origin.

Consider next

$$\psi = \tan^{-1} (r \, d\theta / dr)$$

For the Hyperbolic Spiral

$$\psi = \tan^{-1} (r \, d\theta / dr)$$

$$= \tan^{-1} (- \theta)$$

so that the tangent angle α for this spiral is

$$\alpha = \theta + \tan^{-1} (- \theta)$$

4.0 Conclusion

These three spirals were the subject of a third year mathematical physics assignment given the author by the late Dr. F. Kaempffer at the Department of Physics and Astronomy, University of British Columbia.

This manuscript was written in 1982 but remained lost in the author's library until recently rediscovered.