

The valve DAC

Appendix C. Example noise transfer calculation

To illustrate the noise transfer synthesis method, the required calculations will be done for the third-order topology of Figure 1.

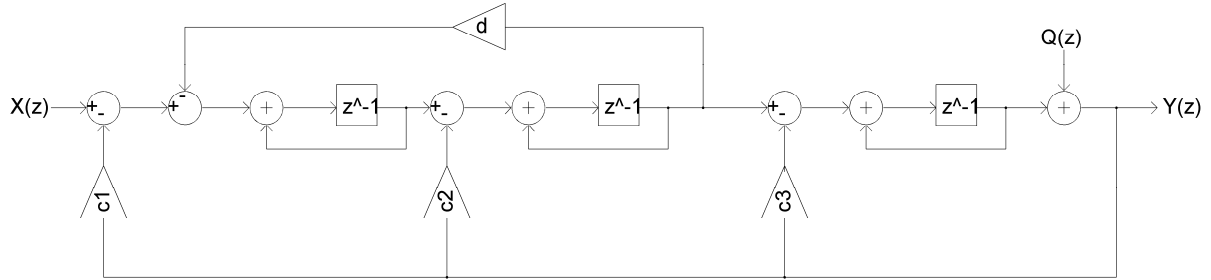


Figure 1 Third-order sigma-delta modulator with the quantizer replaced with a noise addition point

The diagram can be simplified when the local loops are analysed first. The most basic local feedback loop is depicted in Figure 2.

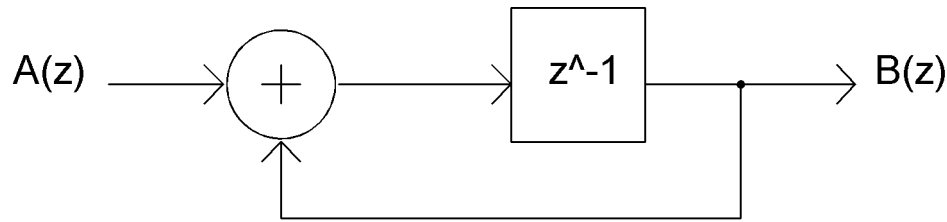


Figure 2 The simplest local feedback loop: an accumulator / integrator

From inspection of the system diagram, we have

$$B(z) = z^{-1}(A(z) + B(z)) \Leftrightarrow B(z)(1 - z^{-1}) = z^{-1}A(z) \Rightarrow \frac{B(z)}{A(z)} = \frac{1}{z - 1}$$

Using this equation, the local resonator loop involving coefficient d can be drawn as in Figure 3.

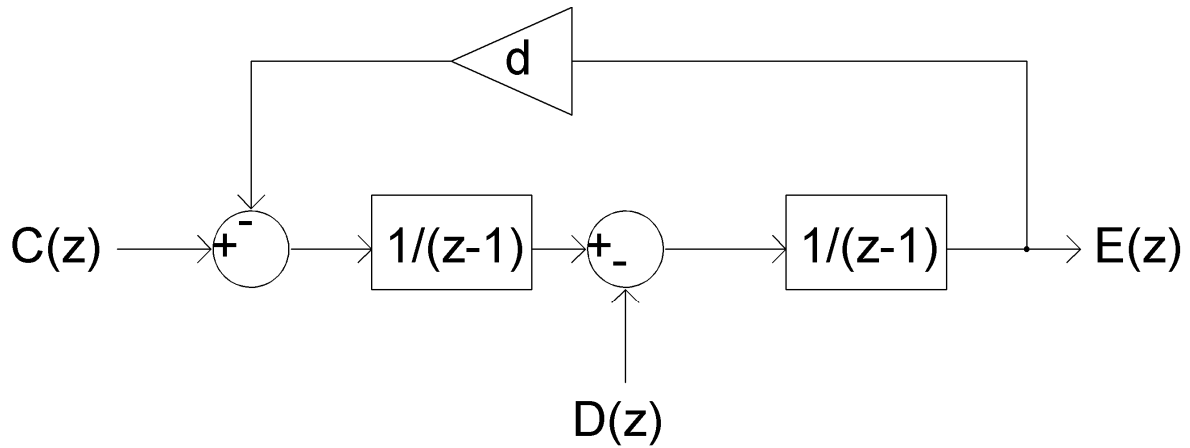


Figure 3 The resonator loop

Hence,

$$E(z) = -\frac{1}{z-1} D(z) + \frac{1}{(z-1)^2} C(z) - d \frac{1}{(z-1)^2} E(z)$$

$$E(z) \left(1 + d \frac{1}{(z-1)^2} \right) = -\frac{1}{z-1} D(z) + \frac{1}{(z-1)^2} C(z)$$

$$E(z) \left((z-1)^2 + d \right) = -(z-1) D(z) + C(z)$$

$$E(z) = -\frac{z-1}{(z-1)^2 + d} D(z) + \frac{1}{(z-1)^2 + d} C(z)$$

The gain of the resonator goes to infinity for $z = 1 \pm j\sqrt{d}$. For $d > 0$, these locations are outside the unit circle, implying that the resonator is unstable and the whole sigma-delta is chaotic. In most practical cases d is chosen $\ll 1$, which means that the sigma-delta is only slightly chaotic. (The poles would have ended up exactly on the unit circle if there had been only a single unit delay in the local loop.) A pole at $z = 1 + j\sqrt{d}$ causes a resonance at

$$f = \frac{1}{2\pi} f_s \arctan(\sqrt{d}), \text{ this being the frequency for which } e^{j2\pi f T} \text{ gets closest to the pole.}$$

The whole loop can now be simplified to Figure 4.

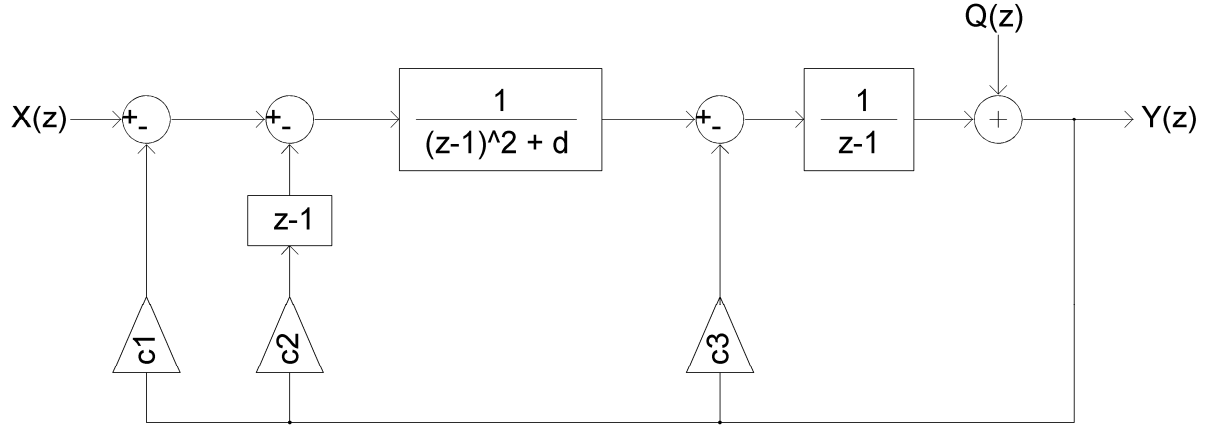


Figure 4 Simplified diagram of the complete third-order loop

Hence,

$$Y(z) = Q(z) - c_3 \frac{1}{z-1} Y(z) - c_2 \frac{1}{(z-1)^2 + d} Y(z) - c_1 \frac{1}{(z-1)((z-1)^2 + d)} Y(z) + \frac{1}{(z-1)((z-1)^2 + d)} X(z)$$

$$Y(z)(z-1)((z-1)^2 + d) = Q(z)(z-1)((z-1)^2 + d) - c_3((z-1)^2 + d)Y(z) - c_2(z-1)Y(z) - c_1Y(z) + X(z)$$

$$Y(z) = Q(z) \frac{(z-1)((z-1)^2 + d)}{c_1 + c_2(z-1) + c_3((z-1)^2 + d) + (z-1)((z-1)^2 + d)} + X(z) \frac{1}{c_1 + c_2(z-1) + c_3((z-1)^2 + d) + (z-1)((z-1)^2 + d)}$$

By definition, the noise transfer function is the transfer from $Q(z)$ to $Y(z)$:

$$NTF(z) = \frac{(z-1)((z-1)^2 + d)}{c_1 + c_2(z-1) + c_3((z-1)^2 + d) + (z-1)((z-1)^2 + d)}$$

while the signal transfer function is the transfer from $X(z)$ to $Y(z)$:

$$STF(z) = \frac{1}{c_1 + c_2(z-1) + c_3((z-1)^2 + d) + (z-1)((z-1)^2 + d)}$$

The signal transfer function is an all-pole low-pass filter transfer that can be quite well-behaved when the pole positions are chosen well. This is an advantage of a sigma-delta modulator structure with multiple feedback paths, as opposed to one with a single feedback and with feedforward structures.

The noise transfer function is a high-pass filter transfer with zeroes at $z = 1$ and at $z = 1 \pm j\sqrt{d}$, the values of z that make the numerator zero. In order to minimize the total noise over the band of interest, d is usually chosen such that the resulting dip in the quantization noise occurs somewhere in the upper part of the band of interest.

A good choice for the pole positions could be Butterworth positions, as they result in a well-behaved signal transfer function and in almost no peaking of the noise transfer function. In the s domain, Butterworth poles lie equally spaced on a semi-circle in the left half plane. For a

third-order filter with radial bandwidth ω_c , this means that the s -domain poles are at $-\omega_c$ and at $-0.5 \omega_c \pm 0.5 \sqrt{3} j \omega_c$.

There are many ways to translate s -domain poles and transfers to z -domain equivalents, each with their own disadvantages, but as long as the sample rate is much greater than the frequencies of interest, they all work quite well. We can therefore take the simplest approach

$$p_z = \exp(p_s T)$$

where T is the sample period time.

Having established the desired pole positions in the z domain and the desired value of d , all that remains is to choose c_1 , c_2 and c_3 such that the poles end up where we want them to be.

The denominator polynomial of $NTF(z)$ can be rewritten as

$$(z-1)^3 + c_3(z-1)^2 + (c_2 + d)(z-1) + (c_1 + c_3 d)$$

If the desired z -domain pole positions are called p_1 , p_2 and p_3 , the desired denominator polynomial is

$$\begin{aligned} (z-p_1)(z-p_2)(z-p_3) = \\ (z-1-(p_1-1))(z-1-(p_2-1))(z-1-(p_3-1)) = \\ (z-1)^3 + (3-p_1-p_2-p_3)(z-1)^2 + ((p_2-1)(p_3-1) + (p_1-1)(p_3-1) + (p_1-1)(p_2-1))(z-1) \\ - (p_1-1)(p_2-1)(p_3-1) \end{aligned}$$

Equating the desired and actual denominator polynomials results in

$$c_3 = 3 - p_1 - p_2 - p_3$$

$$c_2 = -d + (p_2-1)(p_3-1) + (p_1-1)(p_3-1) + (p_1-1)(p_2-1)$$

$$c_1 = -c_3 d - (p_1-1)(p_2-1)(p_3-1)$$

One issue I have quietly swept under the carpet is how to choose the radial bandwidth ω_c of the poles. I simply iteratively changed the bandwidth until the out-of-band gain of $NTF(z)$ was about 1.5 and fine-tuned it after checking transient simulation results (transient simulations were done by writing a simple Pascal program that mimics the sigma-delta modulator). A lower ω_c results in less out-of-band noise gain, in less effective noise shaping and in a larger stable input signal range.

If everything is calculated correctly and the sigma-delta modulator is used in its stable region, the correspondence between the desired noise transfer function and the simulated shaped quantization noise spectrum is usually quite good for a properly dithered multibit sigma-delta. It is not so good for a single-bit sigma-delta modulator. This is the price we pay for modelling a grossly nonlinear system with a simplistic linearized model.

Doing a similar calculation for a seventh-order sigma-delta modulator is left as an exercise to the reader.