

THE THEOEY OF SOUND

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Extracts:

<u>Paragraph</u>	<u>Title</u>
263	Curved Pipe
264	Branched Pipes
265/266	Variable Section

263. Hitherto we have supposed the pipe to be straight, but it will readily be anticipated that, when the cross section is small and does not vary in area, straightness is not a matter of importance. Conceive a curved axis of x running along the middle of the pipe, and let the constant section perpendicular to this axis be S . When the greatest diameter of S is very small in comparison with the wave-length of the sound, the velocity-potential ϕ becomes nearly invariable over the section; applying Green's theorem to the space bounded by the interior of the pipe and by two cross sections, we get

$$\iiint \nabla^2 \phi \, dV = S \cdot \Delta \left(\frac{d\phi}{dx} \right).$$

Now by the general equation of motion

$$\iiint \nabla^2 \phi \, dV = \frac{1}{a^2} \iiint \ddot{\phi} \, dV = \frac{1}{a^2} \frac{d^2}{dt^2} \iiint \phi \, dV = \frac{S}{a^2} \frac{d^2}{dt^2} \int \phi \, dx,$$

and in the limit, when the distance between the sections is made to vanish,

$$\int \phi \, dx = \phi \, dx, \quad \Delta \left(\frac{d\phi}{dx} \right) = \frac{d^2 \phi}{dx^2} \, dx;$$

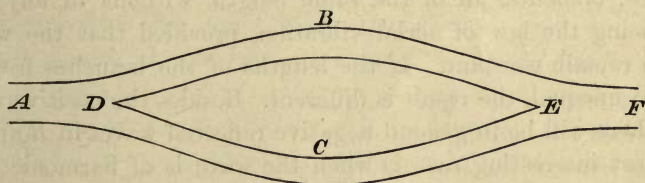
so that

$$\frac{d^2 \phi}{dt^2} = a^2 \frac{d^2 \phi}{dx^2} \dots \dots \dots (1),$$

shewing that ϕ depends upon x in the same way as if the pipe were straight. By means of equation (1) the vibrations of air in curved pipes of uniform section may be easily investigated, and the results are the rigorous consequences of our fundamental equations (which take no account of friction), when the section is supposed to be infinitely small. In the case of thin tubes such as would be used in experiment, they suffice at any rate to give a very good representation of what actually happens.

264. We now pass on to the consideration of certain cases of connected tubes. In the accompanying figure AD represents a thin pipe, which divides at D into two branches DB, DC . At E the branches reunite and form a single tube EF . The sections of the single tubes and of the branches are assumed to be uniform as well as very small.

Fig. 55.



In the first instance let us suppose that a positive wave of arbitrary type is advancing in A . On its arrival at the fork D , it will give rise to positive waves in B and C , and, unless a certain condition be satisfied, to a negative reflected wave in A . Let the potential of the positive waves be denoted by f_A, f_B, f_C, f being in each case a function of $x - at$; and let the reflected wave be $F(x + at)$. Then the conditions to be satisfied at D are first that the pressures shall be the same for the three pipes, and secondly that the whole velocity of the fluid in A shall be equal to the sum of the whole velocities of the fluid in B and C . Thus, using A, B, C to denote the areas of the sections, we have, § 244,

$$\left. \begin{aligned} f'_A - F' &= f'_B = f'_C \\ A(f'_A + F') &= Bf'_B + Cf'_C \end{aligned} \right\} \dots\dots\dots(1);$$

whence

$$F' = \frac{B + C - A}{B + C + A} f'_A \dots\dots\dots(2),$$

$$f'_B = f'_C = \frac{2A}{B + C + A} f'_A \dots\dots\dots(3)^1.$$

¹ These formulæ, as applied to determine the reflected and refracted waves at the junction of two tubes of sections $B + C$, and A respectively, are given by

It appears that f_B and f_C are always the same. There is no reflection, if

$$B + C = A \dots\dots\dots(4),$$

that is, if the combined sections of the branches be equal to the section of the trunk; and, when this condition is satisfied,

$$f_B = f_C = f_A \dots\dots\dots(5).$$

The wave then advances in B and C exactly as it would have done in A , had there been no break. If the lengths of the branches between D and E be equal, and the section of F be equal to that of A , the waves on arrival at E combine into a wave propagated along F , and again there is no reflection. The division of the tube has thus been absolutely without effect; and since the same would be true for a negative wave passing from F to A , we may conclude generally that a tube may be divided into two, or more, branches, all of the same length, without in any way influencing the law of aërial vibration, provided that the whole section remain constant. If the lengths of the branches from D to E be unequal, the result is different. Besides the positive wave in F , there will be in general negative reflected waves in B and C . The most interesting case is when the wave is of harmonic type and one of the branches is longer than the other by a multiple of $\frac{1}{2}\lambda$. If the difference be an *even* multiple of $\frac{1}{2}\lambda$, the result will be the same as if the branches were of equal length, and no reflection will ensue. But suppose that, while B and C are equal in section, one of them is longer than the other by an *odd* multiple of $\frac{1}{2}\lambda$. Since the waves arrive at E in opposite phases, it follows from symmetry that the positive wave in F must vanish, and that the pressure at E , which is necessarily the same for all the tubes, must be constant. The waves in B and C are thus reflected as from an open end. That the conditions of the question are thus satisfied may also be seen by supposing a barrier taken across the tube F in the neighbourhood of E in such a way that the tubes B and C communicate without a change of section. The wave in each tube will then pass on into the other without interruption, and the pressure-variation at E , being the resultant of equal and opposite components, will vanish. This being so, the barrier may be removed without altering the conditions, and no wave will be propagated along F , whatever its section may be. The arrange-

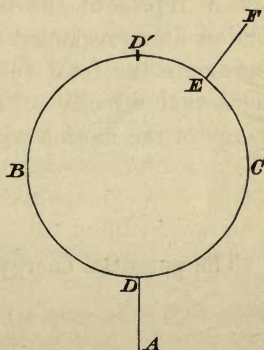
Poisson, *Mém. de l'Institut*, t. II. p. 305, 1819. The reader will not forget that both diameters must be small in comparison with the wave-length.

ment now under consideration was invented by Herschel, and has been employed by Quincke and others for experimental purposes,—an application that we shall afterwards have occasion to describe. The phenomenon itself is often referred to as an example of interference, to which there can be no objection, but the same cannot be said when the reader is led to suppose that the positive waves neutralise each other in F , and that there the matter ends. It must never be forgotten that there is no loss of energy in interference, but only a different distribution; when energy is diverted from one place, it reappears in another. In the present case the positive wave in A conveys energy with it. If there is no wave along F , there are two possible alternatives. Either energy accumulates in the branches, or else it passes back along A in the form of a negative wave. In order to see what really happens, let us trace the progress of the waves reflected back at E .

These waves are equal in magnitude and start from E in opposite phases; in the passage from E to D one has to travel a greater distance than the other by an odd multiple of $\frac{1}{2}\lambda$; and therefore on arrival at D they will be in complete accordance. Under these circumstances they combine into a single wave, which travels negatively along A , and there is no reflection. When the negative wave reaches the end of the tube A , or is otherwise disturbed in its course, the whole or a part may be reflected, and then the process is repeated. But however often this may happen there will be no wave along F , unless by accumulation, in consequence of a coincidence of periods, the vibration in the branches becomes so great that a small fraction of it can no longer be neglected.

Or we may reason thus. Suppose the tube F cut off by a barrier as before. The motion in the ring being due to forces acting at D is necessarily symmetrical with respect to D , and D' —the point which divides $DBCD$ into equal parts. Hence D' is a node, and the vibration is stationary. This being the case, at a point E distant $\frac{1}{4}\lambda$ from D' on either side, there must be a loop; and if the barrier be removed there will still be no tendency to produce vibration in F . If the perimeter of the ring be a multiple of λ , there may be

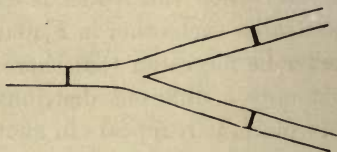
Fig. 56.



vibration within it of the period in question, independently of any lateral openings.

Any combination of connected tubes may be treated in a similar manner. The general principle is that at any junction a space can be taken large enough to include all the region through which the want of uniformity affects the law of the waves, and yet so small that its longest dimension may be neglected in comparison with λ . Under these circumstances the fluid within the space in question may be treated as if the wave-length were infinite, or the fluid itself incompressible, in which case its velocity-potential would satisfy $\nabla^2\phi = 0$, following the same laws as electricity.

Fig. 57.



265. When the section of a pipe is variable, the problem of the vibrations of air within it cannot generally be solved. The case of conical pipes will be treated on a future page. At present we will investigate an approximate expression for the pitch of a nearly cylindrical pipe, taking first the case where both ends are closed. The method that will be employed is similar to that used for a string whose density is not quite constant, §§ 91, 140, depending on the principle that the period of a free vibration fulfils the stationary condition, and may therefore be calculated from the potential and kinetic energies of any hypothetical motion not departing far from the actual type. In accordance with this plan we shall assume that the velocity normal to any section S is constant over the section, as must be very nearly the case when the variation of S is slow. Let X represent the total transfer of fluid at time t across the section at x , reckoned from the equilibrium condition; then \dot{X} represents the total velocity of the current, and $\dot{X} \div S$ represents the actual velocity of the particles of fluid, so that the kinetic energy of the motion within the tube is expressed by

$$T = \frac{1}{2} \rho \int \frac{\dot{X}^2}{S} dx \dots\dots\dots(1).$$

The potential energy § 245 (12) is expressed in general by

$$V = \frac{1}{2} a^2 \rho \iiint s^2 dV,$$

or, since $dV = Sdx$, by

$$V = \frac{1}{2} a^2 \rho \int S s^2 dx \dots\dots\dots (2).$$

Again, by the condition of continuity,

$$-s = \frac{1}{S} \frac{dX}{dx} \dots\dots\dots (3),$$

and thus

$$V = \frac{1}{2} a^2 \rho \int \frac{1}{S} \left(\frac{dX}{dx} \right)^2 dx \dots\dots\dots (4).$$

If we now assume for X an expression of the same form as would obtain if S were constant, viz.

$$X = \sin \frac{\pi x}{l} \cos nt \dots\dots\dots (5),$$

we obtain from the values of T and V in (1) and (4),

$$n^2 = \frac{a^2 \pi^2}{l^2} \int_0^l \cos^2 \frac{\pi x}{l} \frac{dx}{S} \div \int_0^l \sin^2 \frac{\pi x}{l} \frac{dx}{S} \dots\dots\dots (6),$$

or, if we write $S = S_0 + \Delta S$ and neglect the square of ΔS ,

$$n^2 = \frac{a^2 \pi^2}{l^2} \left\{ 1 - 2 \int_0^l \cos \frac{2\pi x}{l} \frac{\Delta S}{S_0} \frac{dx}{l} \right\} \dots\dots\dots (7).$$

The result may be expressed conveniently in terms of Δl , the correction that must be made to l in order that the pitch may be calculated from the ordinary formula, as if S were constant. For the value of Δl we have

$$\Delta l = \int_0^l \cos \frac{2\pi x}{l} \frac{\Delta S}{S_0} dx \dots\dots\dots (8).$$

The effect of a variation of section is greatest near a node or near a loop. An enlargement of section in the first case lowers the pitch, and in the second case raises it. At the points midway between the nodes and loops a slight variation of section is without effect. The pitch is thus decidedly altered by an enlargement or contraction near the middle of the tube, but the influence of a slight conicality would be much less.

The expression for Δl given by (8) is applicable as it stands to the gravest tone only; but we may apply it to the m^{th} tone of the harmonic scale, if we modify it by the substitution of $\cos(2m\pi x/l)$ for $\cos(2\pi x/l)$.

In the case of a tube *open* at both ends (5) is replaced by

$$X = \cos \frac{\pi x}{l} \cos nt \dots \dots \dots (9),$$

which leads to

$$\Delta l = - \int_0^l \cos \frac{2\pi x}{l} \frac{\Delta S}{S_0} dx \dots \dots \dots (10),$$

instead of (8). The pitch of the sound is now raised by an enlargement at the ends, or by a contraction at the middle, of the tube; and, as before, it is unaffected by a slight general conicality (§ 281).

266. The case of progressive waves moving in a tube of variable section is also interesting. In its general form the problem would be one of great difficulty; but where the change of section is very gradual, so that no considerable alteration occurs within a distance of a great many wave-lengths, the principle of energy will guide us to an approximate solution. It is not difficult to see that in the case supposed there will be no sensible reflection of the wave at any part of its course, and that therefore the energy of the motion must remain unchanged¹. Now we know, § 245, that for a given area of wave-front, the energy of a train of simple waves is as the square of the amplitude, from which it follows that as the waves advance the amplitude of vibration varies inversely as the square root of the section of the tube. In all other respects the type of vibration remains absolutely unchanged. From these results we may get a general idea of the action of an ear-trumpet. It appears that according to the ordinary approximate equations, there is no limit to the concentration of sound producible in a tube of gradually diminishing section.

The same method is applicable, when the density of the medium varies slowly from point to point. For example, the amplitude of a sound-wave moving upwards in the atmosphere may be determined by the condition that the energy remains unchanged. From § 245 it appears that the amplitude is inversely as the square root of the density².

¹ *Phil. Mag.* (5) i. p. 261, 1876.

² A delicate question arises as to the ultimate fate of sonorous waves propagated upwards. It should be remarked that in rare air the deadening influence of viscosity is much increased.