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The Planck Constant of Action  $h_A$

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**Highlights**

- The dimension of angle is included in the value of the Planck constant  $h$ .
- Consequently,  $h$  gets the units of action/angle.
- In quantum mechanical theory we need  $h$  with dimensions of action.
- We introduce “the Planck constant of action”  $hA$  or  $hA = hA/2\pi$ .
- $h$ , in J s cycle<sup>-1</sup>, and  $hA$ , in J s, have identical numerical values.

Short communication

The Planck Constant of Action  $h_A$ P. R. Bunker<sup>a</sup>, Per Jensen<sup>b,\*</sup><sup>a</sup>*Steeacie Laboratory, National Research Council of Canada, Ottawa, Ontario K1A 0R6, Canada*<sup>b</sup>*Physikalische und Theoretische Chemie, Fakultät für Mathematik und Naturwissenschaften,  
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**Abstract**

We include the dimension of angle in the expression for the value of the Planck constant  $h$ . As a result it has the dimensions of action/angle. However, in the equations of quantum mechanics we need the Planck constant to have the dimensions of action. Thus we introduce *the Planck constant of action*, which we symbolize as  $h_A$ . The constants  $h$  and  $h_A$  have the same numerical values when  $h$  is expressed in J s cycle<sup>-1</sup> and  $h_A$  in J s.

**Keywords:** The Planck constant, action, J s cycle<sup>-1</sup>, J s

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By including the dimension of angle [1–3], it has been established [4–6] that when we express the *value* of the Planck constant in the units J s cycle<sup>-1</sup> we use the symbol  $h$ , and that when we express it in the units J s radian<sup>-1</sup> we use the symbol  $\hbar$ ;  $h$  is *the Planck constant* and  $\hbar$  is commonly called *the reduced Planck constant*. Thus we have

$$h = 6.62\,607\,015 \times 10^{-34} \text{ J s cycle}^{-1} \quad (1)$$

$$= \hbar = [6.62\,607\,015/(2\pi)] \times 10^{-34} \text{ J s radian}^{-1}. \quad (2)$$

The *numerical value* of  $h$ ,  $6.62\,607\,015 \times 10^{-34}$ , comes from Ref. [7]. The numerical values are in the ratio of  $2\pi$  but their units are in the ratio  $1/2\pi$ , so that the

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values are equal [4–6];  $h = \hbar$  as indicated in Eqs. (1) and (2). The symbols  $h$  and  $\hbar$  occur in familiar equations such as  $E = h\nu = \hbar\omega$ , and  $p = h/\lambda$ . The commonly used units of  $\nu$ ,  $\omega$ , and  $\lambda$  are cycle  $s^{-1}$ , radian  $s^{-1}$ , and m cycle $^{-1}$ , respectively. Just as  $h = \hbar$ , so is also  $\nu = \omega$  (see Ref. [5]). The units of  $\hbar$  are those of angular momentum (see Tables 2 and 4 of Ref. [4]).

The Planck constant sometimes appears with the dimensions of action (energy  $\times$  time) rather than of action/angle as in Eqs. (1) and (2) above. As a result there is a need to introduce a new symbol for *the Planck constant of action* and we suggest the symbol  $h_A$  where

$$h_A = 6.62607015 \times 10^{-34} \text{ J s.} \quad (3)$$

We also need *the reduced Planck constant of action*  $\hbar_A$  where

$$\hbar_A = [6.62607015/(2\pi)] \times 10^{-34} \text{ J s.} \quad (4)$$

The need for such an extended notation for the Planck constant has been considered in a somewhat different fashion in Ref. [8].

While it is not true that  $\hbar = h/2\pi$  [4–6], it is true that  $\hbar_A = h_A/2\pi$ . Having different dimensions,  $h$  and  $h_A$  are different constants but they have the same numerical values when the units J s cycle $^{-1}$  and J s are used. With this new notation for the Planck constant when it has the dimensions of action, the fundamental equations of quantum mechanics should be written as

$$i \hbar_A \frac{d\Psi}{dt} = \hat{H} \Psi \quad (5)$$

and

$$-i \hbar_A \frac{d\Psi}{dq} = \hat{p} \Psi \quad (6)$$

in order that they be dimensionally correct. Similarly, the von Klitzing constant [9]  $R_K$  should be written

$$R_K = \frac{h_A}{e^2} \quad (7)$$

for it to be dimensionally correct. Dirac [10] introduced what we now call  $\hbar_A$  (which he simply called  $\hbar$ ) as a new constant, distinct from the Planck constant, with the

dimension of action (see p. 87 of Dirac's book [11]); this constant is often called *the Dirac constant*.

We pointed out above that  $h$  and  $h_A$  are different constants with different units, but they are intimately related since they have the same numerical values when expressed in the units  $\text{J s cycle}^{-1}$  and  $\text{J s}$ , respectively. Equivalently, their values are connected by the relation

$$\frac{h_A}{h} = 1.00000000 \text{ cycle} \quad (8)$$

where the eight decimal places reflect the precision of the measurement of  $h$  (see Ref. [12] and references therein). Quincey and Burrows [8] have introduced the notation  $\theta_{\text{rev}} = 1 \text{ cycle}$  for the ratio  $h_A/h$ .  $h$  was first introduced by Planck [13] in the expression  $E = h\nu$  for the energy  $E$  of a quantized black-body oscillator with frequency  $\nu$ . We find  $h_A$ , or rather  $\hbar_A = h_A/2\pi$ , in the fundamental equations of quantum mechanics given in Eqs. (5) and (6). The reason for the intimate relationship between  $h$  and  $h_A$  is intriguing and requires further investigation.

Treating angles as not being dimensionless, as we do here, leads to problems of consistency when the angles are used in trigonometric and exponential functions. In these functions angles *are* dimensionless with numerical values consistent with radian units.

We are very grateful to Ian M. Mills for many discussions in which he showed us that it is correct to write  $\hbar = h$  and that we need two different Planck constants. We have further benefited from extensive correspondence with Peter J. Mohr, William D. Phillips and Roberto Marquardt.

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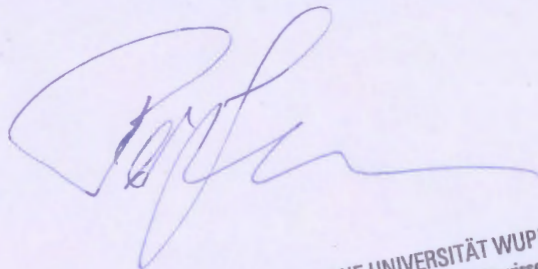
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**Credit Author Statement**

Both authors have contributed equally to the manuscript, with P. R. Bunker writing the original draft and Per Jensen participating in the subsequent editing.

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**Declaration of interests**

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☒ The authors declare the following **financial interests/personal relationships** which may be considered as potential competing interests:

None

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