FDLS Examples

by Greg Berchin and Richard Lyons [January 2007]

The following material provides additional examples of the FDLS algorithm described in the January 2007 IEEE Signal Processing magazine DSP Tips & Tricks column article "Precise Filter Design" by Greg Berchin.

Algebraic Example

Recall the FDLS matrix expression

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_M(0) \end{bmatrix} = \begin{bmatrix} -y_1(-1) & \dots & -y_1(-D) & u_1(0) & \dots & u_1(-N) \\ -y_2(-1) & \dots & -y_2(-D) & u_2(0) & \dots & u_2(-N) \\ \vdots & & \vdots & \vdots & & \vdots \\ -y_M(-1) & \dots & -y_M(-D) & u_M(0) & \dots & u_M(-N) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_D \\ b_0 \\ \vdots \\ b_N \end{bmatrix},$$

which we wrote as $Y = X\Theta$.

Each individual element in the Y column vector is of the form $A_m\cos(\phi_m)$, and each element in the X matrix is of the form $A_1\cos(k\omega_1 t_s + \phi_1)$ or $\cos(k\omega_1 t_s)$. Because all of these elements are of the form of a product (Amplitude)[cos(angle)], each element in Y and X is equal to a constant.

Now if, say, $D = 10$ and $N = 9$, then:

$$y_1(0) = A_1\cos(\phi_1)$$

is the first element of the Y column vector and

$$[-y_1(-1) \ \dots \ -y_1(-10) \ \ u_1(0) \ \dots \ u_1(-9)]$$

is the top row of the X matrix expression where

$$-y_1(-1) = -A_1\cos[(-1)\omega_1 t_s + \phi_1] = -A_1\cos(-\omega_1 t_s + \phi_1)$$

$$-y_1(-2) = -A_1\cos[(-2)\omega_1 t_s + \phi_1] = -A_1\cos(-2\omega_1 t_s + \phi_1)$$

...

$$-y_1(-10) = -A_1\cos[(-10)\omega_1 t_s + \phi_1] = -A_1\cos(-10\omega_1 t_s + \phi_1)$$

and

$$u_1(0) = \cos[(0)\omega_1 t_s] = 1$$

$$u_1(-1) = \cos[(-1)\omega_1 t_s] = \cos(-\omega_1 t_s)$$

$$u_1(-2) = \cos[(-2)\omega_1 t_s] = \cos(-2\omega_1 t_s)$$

...

$$u_1(-9) = \cos[(-9)\omega_1 t_s] = \cos(-9\omega_1 t_s).$$

So the top row of the X matrix looks like:

$$[-A_1\cos(-\omega_1 t_s + \phi_1) \quad -A_1\cos(-2\omega_1 t_s + \phi_1) \ \ldots\ -A_1\cos(-10\omega_1 t_s + \phi_1) \quad 1 \quad \cos(-\omega_1 t_s)$$
$$\cos(-2\omega_1 t_s) \ \ldots\ \cos(-9\omega_1 t_s)].$$

The second row of the X matrix looks like:

$$[-A_2\cos(-\omega_2 t_s + \phi_2) \quad -A_2\cos(-2\omega_2 t_s + \phi_2) \ \ldots\ -A_2\cos(-10\omega_2 t_s + \phi_2) \quad 1 \quad \cos(-\omega_2 t_s)$$
$$\cos(-2\omega_2 t_s) \ \ldots\ \cos(-9\omega_2 t_s)].$$

And so on.

Numerical Example

Here's an example of the above expressions using actual numbers. Suppose we need to approximate the transfer function coefficients for the system whose frequency magnitude and phase response is that shown in Figure E1. Assume that our discrete-system sample rate is 1000 Hz, thus $t_s = 10^{-3}$ seconds, and $N = D = 2$. Also assume $M = 8$ and we have the eight A_1 -to- A_8 magnitude sample values and the eight ϕ_1 -to- ϕ_8 phase samples, shown as dots in Figure E1, available to us as input values to the FDLS algorithm.

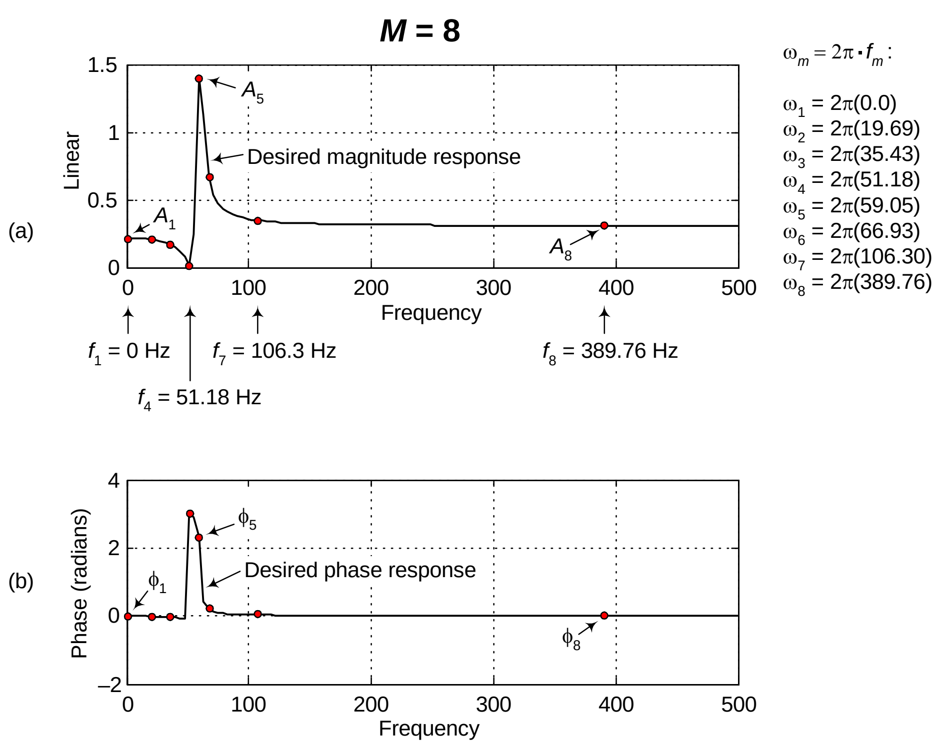

Fig E1

In matrix form, the target analog system parameters are

$$f_m = \begin{bmatrix} 0.0 \\ 19.6850 \\ 35.4331 \\ 51.1811 \\ 59.0551 \\ 66.9291 \\ 106.299 \\ 389.764 \end{bmatrix} \quad \omega_m t_s = \begin{bmatrix} 0.0 \\ 0.1237 \\ 0.2226 \\ 0.3216 \\ 0.3711 \\ 0.4205 \\ 0.6679 \\ 2.449 \end{bmatrix} \quad A_m = \begin{bmatrix} 0.2172 \\ 0.2065 \\ 0.1696 \\ 0.0164 \\ 1.3959 \\ 0.6734 \\ 0.3490 \\ 0.3095 \end{bmatrix} \quad \phi_m = \begin{bmatrix} 0.0 \\ -0.0156 \\ -0.0383 \\ 3.0125 \\ 2.3087 \\ 0.955 \\ 0.0343 \\ 0.0031 \end{bmatrix}$$

where the f_m vector is in Hz, the $\omega_m t_s$ vector is in radians, and $1 \le m \le 8$. The first two elements of the Y vector are:

$$y_1(0) = A_1\cos(\phi_1) = 0.2172\cos(0) = 0.2172.$$

$$y_2(0) = A_2\cos(\phi_2) = 0.2065\cos(-0.0156) = 0.2065.$$

The complete Y vector is:

$$Y = \begin{bmatrix} A_1\cos(\phi_1) \\ A_2\cos(\phi_2) \\ A_3\cos(\phi_3) \\ A_4\cos(\phi_4) \\ A_5\cos(\phi_5) \\ A_6\cos(\phi_6) \\ A_7\cos(\phi_7) \\ A_8\cos(\phi_8) \end{bmatrix} = \begin{bmatrix} 0.2172 \\ 0.2065 \\ 0.1695 \\ -0.0162 \\ -0.9390 \\ 0.6605 \\ 0.3488 \\ 0.3095 \end{bmatrix}$$

The two elements of the "y_1" part of the first row of the X vector are:

$$-y_1(-1) = -A_1\cos(-\omega_1 t_s + \phi_1) = -0.2172\cos(-0 + 0) = -0.2172.$$

$$-y_1(-2) = -A_1\cos(-2\omega_1 t_s + \phi_1) = -0.2172\cos(-0 + 0) = -0.2172.$$

The two elements of the "y_8" part of the eighth row of the X vector are:

$$-y_8(-1) = -A_8\cos(-\omega_8 t_s + \phi_8) = -0.3095\cos(-2.449 + 0.0031) = 0.2376.$$

$$-y_8(-2) = -A_8\cos(-2\omega_8 t_s + \phi_8) = -0.3095\cos(-4.898 + 0.0031) = -0.562.$$

The three elements of the "u_1" part of the first row of the X matrix are:

$u_1(0) = \cos(0) = 1$

$u_1(-1) = \cos(-\omega_1 t_s) = \cos(-0) = 1$

$u_1(-2) = \cos(-2\omega_1 t_s) = \cos(-0) = 1.$

The three elements of the "u_8" part of the eighth row of the X matrix are:

$u_8(0) = \cos(0) = 1$

$u_8(-1) = \cos(-\omega_8 t_s) = \cos(-2.449) = -0.7696$

$u_8(-2) = \cos(-2\omega_8 t_s) = \cos(-4.898) = 0.1845.$

The complete X matrix is:

$$X = \begin{bmatrix} -A_1\cos(-1\omega_1 t_s+\phi_1) & -A_1\cos(-2\omega_1 t_s+\phi_1) & \cos(0) & \cos(-\omega_1 t_s) & \cos(-2\omega_1 t_s) \\ -A_2\cos(-1\omega_2 t_s+\phi_2) & -A_2\cos(-2\omega_2 t_s+\phi_2) & \cos(0) & \cos(-\omega_2 t_s) & \cos(-2\omega_2 t_s) \\ -A_3\cos(-1\omega_3 t_s+\phi_3) & -A_3\cos(-2\omega_3 t_s+\phi_3) & \cos(0) & \cos(-\omega_3 t_s) & \cos(-2\omega_3 t_s) \\ -A_4\cos(-1\omega_4 t_s+\phi_4) & -A_4\cos(-2\omega_4 t_s+\phi_4) & \cos(0) & \cos(-\omega_4 t_s) & \cos(-2\omega_4 t_s) \\ -A_5\cos(-1\omega_5 t_s+\phi_5) & -A_5\cos(-2\omega_5 t_s+\phi_5) & \cos(0) & \cos(-\omega_5 t_s) & \cos(-2\omega_5 t_s) \\ -A_6\cos(-1\omega_6 t_s+\phi_6) & -A_6\cos(-2\omega_6 t_s+\phi_6) & \cos(0) & \cos(-\omega_6 t_s) & \cos(-2\omega_6 t_s) \\ -A_7\cos(-1\omega_7 t_s+\phi_7) & -A_7\cos(-2\omega_7 t_s+\phi_7) & \cos(0) & \cos(-\omega_7 t_s) & \cos(-2\omega_7 t_s) \\ -A_8\cos(-1\omega_8 t_s+\phi_8) & -A_8\cos(-2\omega_8 t_s+\phi_8) & \cos(0) & \cos(-\omega_8 t_s) & \cos(-2\omega_8 t_s) \end{bmatrix}$$

$$= \begin{bmatrix} -0.2172 & -0.2172 & 1.00 & 1.0000 & 1.0000 \\ -0.2045 & -0.994 & 1.00 & 0.9924 & 0.9696 \\ -0.1639 & -0.1502 & 1.00 & 0.9753 & 0.9025 \\ 0.0147 & 0.0117 & 1.00 & 0.9487 & 0.8002 \\ 0.5007 & -0.0059 & 1.00 & 0.939 & 0.7370 \\ -0.6564 & -0.5378 & 1.00 & 0.9129 & 0.6667 \\ -0.2812 & -0.0928 & 1.00 & 0.7851 & 0.2328 \\ 0.2376 & -0.0562 & 1.00 & -0.7696 & 0.1845 \end{bmatrix}.$$

Given the above Y vector and the X matrix, the FDLS algorithm computes the 2nd-order ($N = D = 2$) transfer function coefficients vector $\theta_{M=8}$ as

$$\theta_{M=8} = \begin{bmatrix} -1.8439 \\ 0.9842 \\ 0.3033 \\ -0.5762 \\ 0.3034 \end{bmatrix}.$$

Treated as filter coefficients, we can write vector $\theta_{M=8}$ as:

$a_0 = 1$
$a_1 = -1.8439$
$a_2 = 0.9842$
$b_0 = 0.3033$
$b_1 = -0.5762$
$b_2 = 0.3034$

implemented as the recursive filter network shown in Figure E2.

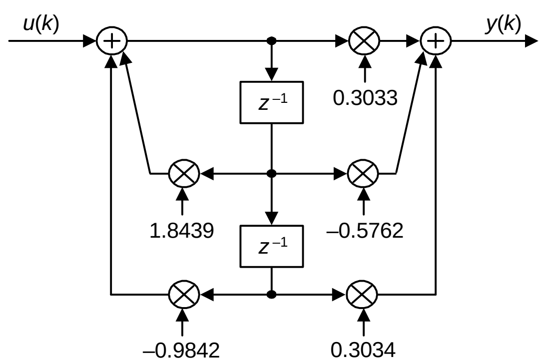

Fig E2

The frequency-domain performance of the filter are the solid red curves shown in Figure E3. There we see that the $\theta_{M=8}$ coefficients provide an accurate approximation to the desired frequency response in Figure E1.

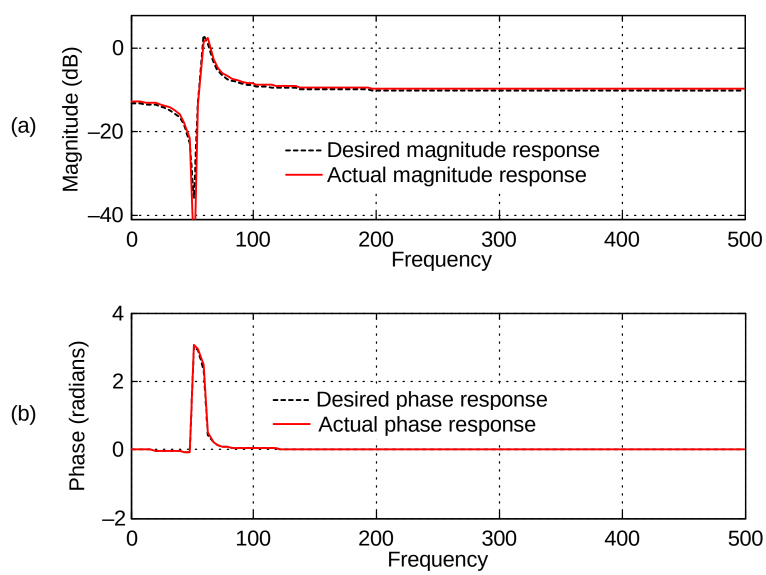

Fig E3