Approximation of the Struve function $\mathbf{H}_1$ occurring in impedance calculations

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The problem of the rigid-piston radiator mounted in an infinite baffle has been studied widely for tutorial as well as for practical reasons. The resulting theory is commonly applied to model a loudspeaker in the audio-frequency range. A special function, the Struve function $\mathbf{H}_1(z)$, occurs in the expressions for the rigid-piston radiator. This Struve function is not readily available in programs such as Matlab or Mathcad, nor in computer languages such as FORTRAN and C. Therefore a simple and effective approximation of $\mathbf{H}_1(z)$ which is valid for all z is developed. Some examples of the application of the Struve function in acoustics are presented. © *2003 Acoustical Society of America.* [DOI: 10.1121/1.1564019]

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I. INTRODUCTION

Struve functions occur in many places in physics and applied mathematics, e.g., in optics as the normalized line-spread function (de Boer *et al.*, 1994), in fluid dynamics (Newman, 1984), and quite prominently in acoustics for impedance calculations as is outlined below. The problem of the rigid-piston radiator mounted in an infinite baffle has been studied widely for tutorial as well as for practical reasons (see, e.g., Greenspan, 1979; Pierce, 1989; Kinsler *et al.*, 1982; Beranek, 1954; Morse and Ingard, 1968). The resulting theory is commonly applied to model a loudspeaker in the audio-frequency range. For a baffled piston the ratio of the force amplitude to the normal velocity amplitude, termed the piston mechanical radiation impedance, is given by

$$Z_m = \frac{-i\omega\rho}{2\pi} \int\int\int\int R^{-1} e^{ikR}\, dx_s\, dy_s\, dx\, dy. \tag{1}$$

Here $R=\sqrt{(x-x_s)^2+(y-y_s)^2}$ is the distance between any two surface points on the piston (x_s, y_s) and (x, y), respectively; ω is the frequency and ρ is the density of air. The integration limits are such that (x_s, y_s) and (x, y) are within the area of the piston. The fourfold integral in Eq. (1), known as the Helmholtz integral, was performed by Rayleigh (1896, §302) and further elaborated in Pierce (1989), with the result

$$Z_m = \rho c \pi a^2 [R_1(2ka) - iX_1(2ka)], \tag{2}$$

where

$$R_1(2ka) = 1 - \frac{2J_1(2ka)}{2ka} \tag{3}$$

and

$$X_1(2ka) = \frac{2\mathbf{H}_1(2ka)}{2ka} \tag{4}$$

are the real and imaginary part of the radiation impedance, respectively. In Eqs. (2)–(4), a is the piston radius, k is the wave number ω/c, c is the speed of sound, J_1 is the first-order Bessel function of the first kind (Abramowitz and Stegun, 1972, §9.1.21), and $\mathbf{H}_1(z)$ is the Struve function of the first kind (Abramowitz and Stegun, 1972, §12.1.6). See Fig. 1 for a plot of $R_1(ka)$ and $X_1(ka)$.

The Struve function is not readily available in programs such as Matlab or Mathcad, nor in computer languages such as FORTRAN and C. High-accuracy expansions and approximations of Struve functions are available in the literature, see for instance Newman (1984) and the references therein, but these are somewhat cumbersome to use since they require separate consideration of small and large z. Also, the accuracy provided by these approximations is far beyond what is actually needed in most acoustical applications. Therefore, in the following an effective and simple approximation of $\mathbf{H}_1(z)$ which is valid for all z is developed.

II. APPROXIMATION OF THE STRUVE FUNCTION $\mathbf{H}_1(z)$

The Struve function $\mathbf{H}_1(z)$ is defined as

$$\mathbf{H}_1(z) = \frac{2z}{\pi} \int_0^1 \sqrt{1-t^2} \sin zt\, dt. \tag{5}$$

There is the power series expansion (Abramowitz and Stegun, 1972, §12.1.5)

$$\mathbf{H}_1(z) = \frac{2}{\pi}\left[\frac{z^2}{1^2 3} - \frac{z^4}{1^2 3^2 5} + \frac{z^6}{1^2 3^2 5^2 7} - \cdots\right]. \tag{6}$$

For the purpose of numerical computation this series is only useful for small values of *z*, *ka*, respectively. Equations (4) and (6) yield for small values of *ka*

$$X_1(ka) \approx \frac{8ka}{3\pi}, \tag{7}$$

which is in agreement with the small *ka* approximation as can be found in the references given earlier (see also Fig. 1). Furthermore, there is the asymptotic result (Abramowitz and Stegun, 1972, §12.1.31, §9.2.2 with $\nu=1$)

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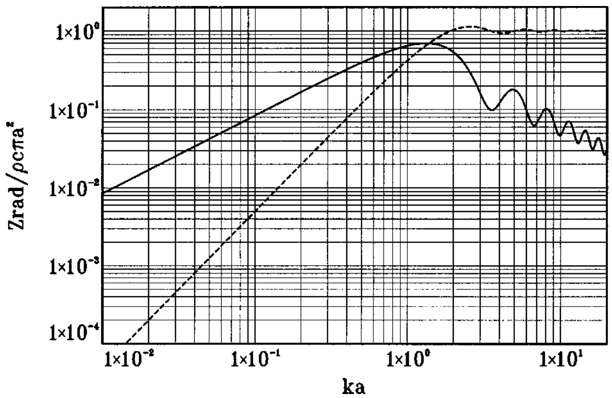

FIG. 1. Real part R_1 (dashed line) and imaginary part X_1 (solid line) of the normalized radiation impedance of a rigid disk with a radius a in an infinite baffle. (On a log log scale.)

$$\mathbf{H}_1(z)=\frac{2}{\pi}-\sqrt{\frac{2}{\pi z}}\cos(z-\pi/4)+O(1/z),\quad z\to\infty,\quad (8)$$

but this is only useful for large values of z. Equation (4) and the first term of Eq. (8) yield for large values of ka

$$X_1(ka)\approx\frac{2}{\pi ka},\quad (9)$$

and this is in agreement with the large ka approximation as can be found in the earlier given references as well.

Below an approximation for all values of ka is developed in which only a limited number of elementary functions is involved. Integrating by parts, the integral in Eq. (5) becomes

$$\mathbf{H}_1(z)=\frac{2}{\pi}\left[1-\int_0^1\frac{t\cos zt}{\sqrt{1-t^2}}dt\right].\quad (10)$$

The integral on the right-hand side of Eq. (10) can be written as

$$\int_0^1\frac{\cos zt}{\sqrt{1-t^2}}dt-\int_0^1\sqrt{\frac{1-t}{1+t}}\cos zt\,dt.\quad (11)$$

Using the integral representation of $J_0(z)$ (Abramowitz and Stegun, 1972, §9.1.20 with $\nu=0$) there results the exact representation

$$\mathbf{H}_1(z)=\frac{2}{\pi}-J_0(z)+\frac{2}{\pi}\int_0^1\sqrt{\frac{1-t}{1+t}}\cos zt\,dt.\quad (12)$$

The function $\sqrt{(1-t)/(1+t)}$ in the remaining integral on the right-hand side of Eq. (12) can be approximated quite well by a linear function of t. Accordingly, there holds

$$\sqrt{\frac{1-t}{1+t}}\approx\hat{c}+\hat{d}t,\quad (13)$$

where $\hat{c}$ and $\hat{d}$ are such that

$$\int_0^1\left|\sqrt{\frac{1-t}{1+t}}-(c+dt)\right|^2dt\quad (14)$$

is minimal for $c=\hat{c}$, $d=\hat{d}$. These $\hat{c}$, $\hat{d}$ are readily obtained by requiring that $\sqrt{(1-t)/(1+t)}-(\hat{c}+\hat{d}t)$ is orthogonal to the functions 1, t on [0, 1], and this yields

$$\hat{c}=7\pi/2-10,\quad \hat{d}=18-6\pi,\quad (15)$$

with minimum squared error in Eq. (14) equal to 3.4×10^{-4}.

There results the approximation

$$\mathbf{H}_1(z)\approx\frac{2}{\pi}-J_0(z)+\left(\frac{16}{\pi}-5\right)\frac{\sin z}{z}+\left(12-\frac{36}{\pi}\right)\frac{1-\cos z}{z^2},\quad (16)$$

with squared approximation error on $[0,\infty)$ equal to 2.2×10^{-4} by Parseval's formula.

It is observed that the right-hand side of Eq. (16) equals $0=\mathbf{H}_1(0)$ for $z=0$. The absolute approximation error in Eq. (16) is plotted in Fig. 2 as a function of z. For the calculation of $\mathbf{H}_1(z)$ the computer program Mathematica has been used. Using Mathematica (V.4.0.2.0) with standard precision results into an anomaly in the region of z between 26 and 30; this anomaly disappears when the standard precision is replaced by a higher one as is done for Fig. 2.

As one sees, the approximation error is small and decently spread-out over the whole z-range, it vanishes for $z=0$, and its maximum value is about 0.005. Replacing $\mathbf{H}_1(z)$ in Fig. 1 by the approximation in Eq. (16) would result in no

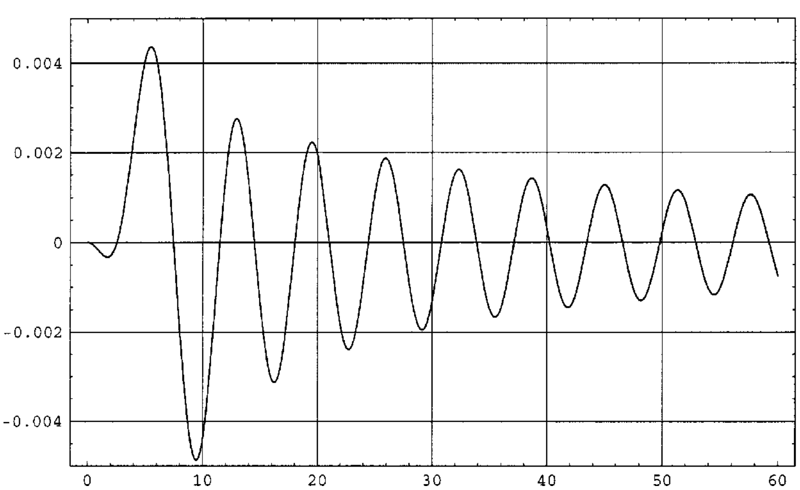

FIG. 2. The absolute error in the approximation of $\mathbf{H}_1(z)$ by Eq. (16).

visible change at all. The maximum relative error appears to be less than 1%, equals 0.1% at $z=0$, and decays to zero for $z\rightarrow\infty$.

It was found that inclusion of a quadratic and a cubic term in the approximation in Eq. (13) resulted in a decrease of the mean squared error, see Eq. (14), by only a factor of 1.5 and 6, respectively. The resulting approximation of $\mathbf{H}_1(z)$ then becomes rather awkward, both in form and numerically, while its accuracy is only marginally improved. Hence the approximation in Eq. (16) seems the best compromise between accuracy and simplicity.

III. EXAMPLE

A prime example of the use of the radiation impedance is the calculation of the radiated acoustic power of a circular piston in an infinite baffle. This is an accurate model (Beranek, 1954) for a loudspeaker with radius a mounted in a large cabinet. The radiated acoustic power is equal to

$$P_a=0.5|V|^2\Re\{Z_m\}, \tag{17}$$

where V is the velocity of the loudspeaker's cone, and $\Re$ denotes real part of. An example of the use of the obtained approximation of $\mathbf{H}_1$ is to calculate a loudspeaker electrical input impedance Z_{in}, which is among other parameters a function of Z_m (see Beranek, 1954; Vanderkooy and Boers, 2002). Using Z_{in}, the time-averaged electrical power delivered to the loudspeaker is calculated as

$$P_e=0.5|I|^2\Re\{Z_{\text{in}}\}, \tag{18}$$

where I is the current fed into the loudspeaker. Finally, the efficiency of a loudspeaker, defined as

$$\eta(ka)=P_a/P_e, \tag{19}$$

which is an important engineering parameter in the field of electro-acoustics, can be calculated. The approximation in Eq. (16) of $\mathbf{H}_1(z)$ is used in Vanderkooy and Boers (2002) to calculate Z_{in} and $\eta(ka)$.

IV. CONCLUSIONS

A simple and effective approximation of the Struve function $\mathbf{H}_1(z)$ for all values of z has been developed using only a limited number of elementary functions. The obtained approximation is in agreement with the small as well as large ka approximations known from the literature, but does not require patchwork formulas, since it is accurate for the whole ka range. The approximation can be used in various fields, with its most prominent engineering application in electro-acoustics.

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