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Calculation of the surface pressure on a vibrating circular stretched membrane in free space

J. H. Streng
Philips Research Laboratories Eindhoven, P. O. Box 80.000, 5600 JA Eindhoven, The Netherlands

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In this article, a method is presented for the calculation of the surface pressure on an edge-clamped vibrating circular stretched membrane in an infinitely extended homogeneous isotropic elastic medium. The coupled differential equations describing the (linear) rotationally symmetric membrane deflection and the (linear) wave propagation in the medium are solved simultaneously by the introduction of a series expansion of the surface pressure. The coefficients in this expansion may be calculated numerically by means of collocation, which is demonstrated for the case of an edge-clamped vibrating circular membrane in air.

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INTRODUCTION

This article is concerned with the analysis of the fluid-loading effects on the vibrational behavior of a circular (loudspeakerlike) membrane in air (although the formal analysis allows an arbitrary medium), excited by a homogeneous force distribution. The fluid loading is usually characterized by a parameter $\epsilon = \lambda\rho_0/(2\pi\rho)$, essentially the ratio of fluid mass density ρ_0 and membrane surface mass density ρ times fluid wavelength λ. In many cases (underwater acoustics, in-air plate vibrations), this parameter is small and the analysis may focus on an asymptotic solution in the limit $\epsilon \ll 1$ ("light fluid loading") whereas, in other cases, an asymptotic solution in the limit $\epsilon \gg 1$ ("heavy fluid loading") may be of interest (see, for example, Refs. 1, 2). In this article, the range of ϵ may very well include unity, which may be referred to as "significant fluid loading." The "membrane-in-air" problem has received attention in the literature (see, for example, Ref. 3), but usually the analysis is restricted to baffled membranes (Huygens–Rayleigh approach) and the farfield region (Fraunhofer zone). The nearfield analysis often has a frequency restriction (very low or very high frequencies).

In this article, we will focus on the unbaffled membrane in free space and give a numerical method for the analysis of the nearfield behavior. In theory, the method is applicable to an arbitrary frequency range (except for the *in vacuo* eigenfrequencies of the membrane in the case of vanishing material damping). The line of work will be as follows: In Sec. I we will examine the basic (mathematical) formulation of the problem, whereas in Sec. II, a power series solution for the surface pressure on the membrane is calculated. In Sec. III we will examine some results obtained for a loudspeakerlike membrane vibrating in air. It must be emphasized that the mathematical analysis in Sec. II is the main subject of this article (more detailed results are reserved for future articles). In Sec. II A, we start the analysis by the introduction of a series expansion (derived by Bouwkamp[4]) for the surface pressure. Next, by means of a "traditional" Green's function technique, we will derive two expressions for the particle movement at the interface membrane medium. One expression results from the Helmholtz equation in free space (comparable to the approach in "baffled-membrane" theory, where the use of the Lamb or Sommerfeld integral leads to a similar "King-like" expression[5]) and the other from the membrane equation of motion, corresponding to an eigenfunction expansion (Secs. II B and C, respectively). It will be shown that a high-frequency approximation can easily be derived (which is a well-known fact,[3] but derived in a different way here). The next step (Sec. II D) is to equate the two expressions, which usually leads to an integral equation (or an integro-differential equation) for the surface pressure, but in our case it leads to an infinite set of equations for the power series coefficients. These coefficients may then be calculated by means of a collocation technique. In Sec. II F the numerical evaluation of the power series coefficients is elaborated. In the results (Sec. III), we will restrict our attention to the low frequency range, as we calculate some results for the case of a loudspeakerlike thin membrane vibrating in air for frequencies less than the second lowest *in vacuo* eigenfrequency of the membrane.

We will only consider the case of rotationally symmetric variations.

I. FORMULATION OF THE PROBLEM

A. The governing equations

We consider a circular stretched membrane of radius a (m), surface mass density $\rho\,(\mathrm{kg/m^2})$, tension $T\,(\mathrm{N/m})$, and negligible thickness and bending stiffness, situated in the x–y plane with its center at the origin. The membrane is clamped along its circumference $r = a$ and embedded in an infinitely extended, homogeneous, nonviscous, isotropic, and elastic medium of density ρ_0 $(\mathrm{kg/m^3})$ and sound velocity c_0 (m/s). We assume that a homogeneous force distribution F $(\mathrm{N/m^2})$ is acting upon the membrane in positive z direction. Assuming a harmonic time dependence $e^{j\omega t}$ and applying a small-signal approach, the governing equations for the membrane deflection $\eta(r)$ in positive z direction and the

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sound pressure $p(r,z)$ in the surrounding medium can be written as

$$T\nabla_2^2\eta(r)+\omega^2\rho\eta(r)-[p_+(r)-p_-(r)]+F=0,\quad r<a,$$
$$\eta(a)=0, \tag{1}$$
$$\nabla_3^2 p(r,z)+k^2p(r,z)=0,$$

where

$$p_+(r)=p(r,0^+),$$
$$p_-(r)=p(r,0^-),$$
$$\nabla_2^2\{\cdot\}=\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\,\partial r}\right)\{\cdot\},$$
$$\nabla_3^2\{\cdot\}=\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\,\partial r}+\frac{\partial^2}{\partial z^2}\right)\{\cdot\},$$
$$k^2=(\omega/c_0)^2.$$

Due to the geometrical symmetry, we may write

$$p(r,z)=-p(r,-z),\qquad p(r,0)=0,\quad r>a. \tag{2}$$

Next, we have the usual coupling constraint

$$\left.\frac{\partial p(r,z)}{\partial z}\right|_{z=0^+}=\left.\frac{\partial p(r,z)}{\partial z}\right|_{z=0^-}=\omega^2\rho_0\eta(r),\quad r<a, \tag{3}$$

which states that the normal component of the particle velocity in the medium equals the membrane velocity at the interface medium–membrane. Finally, from (2) and (3) at $r=a$, we may derive that

$$\left.\frac{\partial p(r,0)}{\partial r}\right|_{r\uparrow a}=0,\qquad \left.\frac{\partial p(a,z)}{\partial z}\right|_{z=0^\pm}=0. \tag{4}$$

Thus, at the edge $r=a$, there is neither outward-radial nor axial particle movement. For reasons of continuity, we conclude that

$$\left.\frac{\partial p_\pm(r)}{\partial r}\right|_{r\uparrow a}=0. \tag{5}$$

Here, we have an essential difference from the free-piston model. Bouwkamp[4] has already proved that the radial particle velocity in a free-piston model will become singular at $r\uparrow a$.

B. The basic mathematical formulation

For a mathematical analysis it is convenient to introduce normalized coordinates r_n and z_n, normalized pressure $q(r_n)$, and membrane deflection $w(r_n)$ according to

$$w(r_n)=\eta(r_na)(T/a^2F),$$
$$q(r_n)=p(r_na,z_na)(T/a^3\omega^2\rho_0F), \tag{6}$$
$$r_n=r/a,$$
$$z_n=z/a.$$

The subscript n will be omitted in the remaining text. Substitution of the normalized variables in Eqs. (1)–(5) leads to the following set of equations:

$$\nabla_2^2 w(r)+\alpha^2w(r)=\gamma^2q_+(r)-1,\quad r<1,$$
$$q_+(r)=q(r,0^+),$$
$$w(1)=0,$$
$$\nabla_3^2q(r,z)+\beta^2q(r,z)=0, \tag{7}$$
$$\left.\frac{\partial q(r,z)}{\partial z}\right|_{z=0^\pm}=w(r),\quad r<1,$$
$$\left.\frac{\partial q_+(r)}{\partial r}\right|_{r\uparrow 1}=0,$$

where $\alpha=a\omega(\rho/T)^{1/2}$ is the normalized *in vacuo* wavenumber in the membrane, $\beta=ka$ is the normalized wavenumber in the medium, and $\gamma=a\omega(2a\rho_0/T)^{1/2}$ is the normalized fluid-loading parameter.

We see that the problem is essentially "controlled" by three parameters: α,β, and γ, which are defined in (7), apart from the scaling factors in (6). These parameters may be converted to another set of three parameters; for example, one may use α, $M=\beta/\alpha$ (the "Mach number") and $\epsilon=\gamma^2/(2\alpha^2\beta)=\lambda\rho_0/(2\pi\rho)$ (the "fluid-loading parameter," which is used by Leppington[2]). Usually the calculations are carried out for (very) small or (very) large values of ϵ. For a loudspeakerlike membrane in air, however, the value of ϵ may very well be in a range including unity, and it is this case that we are interested in.

II. CONSTRUCTION OF A POWER SERIES SOLUTION

A. Introduction of oblate spheroidal coordinates

In order to find a solution $q(r,z)$ to the Helmholtz equation in (7), one usually introduces a suitable transformation of coordinates such that the equation is separable in the new coordinates and the geometry of the problem may be described in coordinate planes. Bouwkamp[4] showed that in the case in question we should introduce oblate spheroidal coordinates according to

$$z=\xi\chi,$$
$$r=(1-\xi^2)^{1/2}(1+\chi^2)^{1/2}, \tag{8}$$
$$1>\xi>-1,\quad \chi>0.$$

The area $r<1$, $z=0^+$ now corresponds to $\chi=0^+$, $0^+<\xi<1$. It can be shown[4] that substitution of (8) into the Helmholtz equation leads to a separable equation, and thus the solution $q(\xi,\chi)$ may be written as

$$q(\xi,\chi)=\sum_i d_i\Xi_i(\xi)X_i(\chi). \tag{9}$$

Next, we introduce the power series

$$\Xi_i(\xi)=\sum_{n=0}^{\infty}c_{ni}\xi^n, \tag{10}$$

which may be shown to converge.[4] As we are merely interested in an odd solution with respect to ξ, we set $c_n=0$ for n even. The corresponding function $q_+(r)$ now reads

$$q_+(\xi)=\sum_i d_i\Xi_i(\xi)X_i(0^+),\quad 0^+<\xi<1,$$

or (11)

$$q_+(r)=\sum_{n=0}^{\infty}a_n(1-r^2)^{n+1/2},\quad r<1.$$

Finally, we use $\partial q_+(r)/\partial r|_{r\uparrow 1}=0$, which leads to $a_0=0$ and $q_+(1)=0$, and arrive at

$$q_+(r) = \sum_{n=1}^{\infty} a_n (1 - r^2)^{n+1/2}, \quad r<1. \tag{12}$$

B. Expression for $w(r)$ resulting from the wave equation

An expression for $w(r)$, the normalized membrane deflection, is obtained using the free-space Green's function $g(r,\varphi,z|r_0,\varphi_0,z_0)$ of the Helmholtz equation in (7). The normalized pressure $q(r,z)$ in terms of the source distribution $q_+(r)$ and the free-space Green's function reads[3]

$$q(r,z) = \int_0^{2\pi} \int_0^1 2q_+(r_0) \times \frac{\partial [g(r,\varphi,z|r_0,\varphi_0,z_0)]}{\partial z_0}\bigg|_{z_0=0} r_0\, dr_0\, d\varphi_0,$$

where

$$g(r,\varphi,z|r_0,\varphi_0,z_0) = \frac{-j}{(4\pi)} \sum_{n=0}^{\infty} \delta_n \cos[n(\varphi - \varphi_0)] \times \int_0^{\infty} \left(\frac{\mu}{\sigma}\right) J_n(\mu r_0) J_n(\mu r) e^{-j\sigma|z_0 - z|}\, d\mu, \tag{13}$$

$$\delta_n = \begin{cases} 1, & n=0, \\ 2, & n\neq 0, \end{cases}$$

$$\sigma = \begin{cases} (\beta^2 - \mu^2)^{1/2}, & \beta>\mu, \\ -j(\mu^2 - \beta^2)^{1/2}, & \beta<\mu. \end{cases}$$

From (13) it is clear that the φ integration may be carried out, leaving only the $n=0$ term. The result (14) is the "unbaffled" version of the Lamb or Sommerfeld integral:

$$q(r,z) = \int_0^1 q_+(r_0) \int_0^{\infty} J_0(\mu r) J_0(\mu r_0) \times e^{-j\sigma z} \mu\, d\mu\, r_0\, dr_0, \quad z>0. \tag{14}$$

Integral (14) reduces, not surprisingly, to a double Hankel transform for $z\downarrow 0$. Next, we use the coupling constraint from (7),

$$\frac{\partial q(r,z)}{\partial z}\bigg|_{z=0^\pm} = w(r), \quad r<1, \tag{15}$$

and arrive at

$$w(r) = -j \int_0^{\infty} J_0(\mu r)\sigma\mu\, d\mu \int_0^1 q_+(r_0) \times J_0(\mu r_0) r_0\, dr_0, \quad r<1. \tag{16}$$

Finally, we substitute the power series (12) into (16) and use the identity[6]

$$\int_0^1 r(1 - r^2)^{n+1/2} J_0(sr)dr = 2^{n+1/2}\Gamma\left(n + \frac{3}{2}\right)\left(\frac{1}{s}\right)^{n+3/2} J_{n+3/2}(x), \tag{17}$$

which yields

$$w(r) = -j \sum_{n=1}^{\infty} a_n 2^{n+1/2}\Gamma\left(n + \frac{3}{2}\right) I_n(r,\beta), \quad r<1, \tag{18}$$

where

$$I_n(r,\beta) = \int_0^{\infty} \left(\frac{1}{\mu}\right)^{n+1/2} \sigma J_0(\mu r) J_{n+3/2}(\mu) d\mu.$$

Substitution of a large value for β into (18) yields a high-frequency approximation. For $\beta \gg 1$, the integral $I_n(r,\beta)$ may be approximated by[7]:

$$\beta \int_0^{\infty} \left(\frac{1}{\mu}\right)^{n+1/2} J_0(\mu r) J_{n+3/2}(\mu) d\mu = \beta (\tfrac{1}{2})^{n+1/2}(1 - r^2)^{n+1/2}/\Gamma(n + \tfrac{3}{2}), \quad 1>r>0. \tag{19}$$

Using this result in (18), we get

$$w(r) \simeq -j\beta \sum_{n=1}^{\infty} a_n (1 - r^2)^{n+1/2} = -j\beta q_+(r), \quad \beta \gg 1. \tag{20}$$

Substitution of the scaling factors (6) and $\beta = ka$ into (20) yields the specific acoustic impedance z_a:

$$z_a = p_+(r)/[j\rho_0 c_0 \omega \eta(r)] = 1, \quad ka \gg 1, \tag{21}$$

which is a result already reported by Morse and Ingard,[3] but derived in a slightly different way here.

C. Expression for $w(r)$ resulting from the membrane equation

An expression for $w(r)$ starting at the membrane equation (22) is obtained in the same way as in the previous section, using a suitable Green's function, which results in an eigenfunction expansion. We have the membrane equation

$$\nabla_2^2 w(r) + \alpha^2 w(r) = \gamma^2 q_+(r) - 1, \quad r<1, \qquad w(1) = 0, \tag{22}$$

and its solution[3]

$$w(r) = \int_0^1 [\gamma^2 q_+(r_0) - 1] G(r|r_0) r_0\, dr_0, \quad r<1,$$

where

$$G(r|r_0) = \sum_{m=1}^{\infty} \frac{2J_0(j_{0m} r) J_0(j_{0m} r_0)}{J_1(j_{0m})^2(\alpha^2 - j_{0m}^2)}, \tag{23}$$

and j_{0m} are the zeros of $J_0(r)$.

Using again the power series expansion (12) and the integral (16), the expression for $w(r)$ in (23) may be calculated straightforwardly:

$$w(r) = -\Phi(r,\alpha) + \sum_{n=1}^{\infty} a_n 2^{n+1/2} \times \Gamma(n + \tfrac{3}{2}) \Psi_n(r,\alpha,\gamma), \quad r<1,$$

where

$$\Psi_n(r,\alpha,\gamma) = \sum_{m=1}^{\infty} \frac{2\gamma^2 J_0(j_{0m} r) J_{n+3/2}(j_{0m})}{J_1(j_{0m})^2(\alpha^2 - j_{0m}^2) j_{0m}^{n+3/2}}, \tag{24}$$

$$\Phi(r,\alpha) = \sum_{m=1}^{\infty} \frac{2J_0(j_{0m} r)}{J_1(j_{0m}) j_{0m}(\alpha^2 - j_{0m}^2)},$$

where we have used[7]

$$\int_0^1 J_0(j_{0m} r_0) r_0\, dr_0 = \frac{J_1(j_{0m})}{j_{0m}}. \tag{25}$$

The function $\Phi(r,\alpha)$ in the right-hand side of (24), which corresponds to the forced *in vacuo* vibration, may be written as $[1-J_0(\alpha r)/J_0(\alpha)]/\alpha^2$, as is easily checked by substitution into (22). Clearly, the eigenfunction expansion method fails for the *in vacuo* eigenfrequencies where $\alpha = j_{0m}$. This problem is discussed by Leppington[2] and we will not deal with it here [for a loudspeakerlike membrane, the resonant modes are usually damped; this (viscous) damping will enter the membrane equation as a nonzero imaginary part of α].

D. Calculation of the power series coefficients

If we equate the right-hand sides of (18) and (24), we obtain the following expression:

$$\sum_{n=1}^{\infty} a_n {}^{n+1/2}\Gamma\left(n+\frac{3}{2}\right)[jI_n(r,\beta)+\Psi_n(r,\alpha,\gamma)]$$
$$=\Phi(r,\alpha), \quad r<1. \tag{26}$$

Any number of coefficients a_n, say N, may be calculated from (26) by means of collocation: Substitution of N different values of r into (26) will give a set of N linear equations for the coefficients a^n, $n=(1...N)$. We see that a fast convergence of the functions $\Psi_n(r,\alpha,\gamma)$ and $\Phi(r,\alpha)$ is ensured by the presence of a factor j_{0m}^3 in the denominators, provided that $\alpha \neq j_{0i}$. In Sec. II F we will examine the behavior of the function $I_n(r,\beta)$.

E. Calculation of the nearfield

For a high-frequency calculation, we may use the approximation (20) and the membrane equation of motion (22) to calculate the membrane deflection and the surface pressure. The solution is easily found to be

$$w(r) = -[1-J_0(\sigma r)/J_0(\sigma)]/\sigma^2, \tag{27}$$

where $\sigma^2 = (\alpha^2 - j)\gamma^2/\beta$.

Once the surface pressure is calculated from (26), we may calculate the pressure throughout the medium from (14) and the property of symmetry $q(r,z) = -q(r,-z)$. We notice that we do not encounter any numerical problems due to the behavior of the term $e^{-j\alpha z}$ in (14). A field quantity that may be of special interest in the analysis of the fluid-loading effects is the time-averaged power flow or (vector) intensity $\mathbf{I}(r,z)$. The intensity is calculated from half the real part of the product of pressure and complex conjugate of the particle velocity; the particle velocity may, as usual, be calculated from the gradient of the pressure. Using normalized variables (6), the intensity may be written as

$$\mathbf{I}(r,z) = C\,\mathrm{Re}\left[-jq(r,z)\left(\overline{\frac{\partial q(r,z)}{\partial r}\hat{a}_r, \frac{\partial q(r,z)}{\partial z}\hat{a}_z}\right)\right], \tag{28}$$

where C is a positive constant, $\hat{a}_r$, $\hat{a}_z$ denote unit vectors in r, z directions, respectively, and the horizontal bar denotes complex conjugation. The pressure gradient in (28) may be calculated from (14) without numerical problems.

F. Numerical evaluation

The numerical evaluation of Eq. (26) yields no problems except the integral in I_n which has a strongly oscillating integrand, which may well lead to severe numerical difficulties. In order to avoid this problem, we rewrite the integral

$$I_n(r,\beta) = (1/\beta)^{n-3/2} I_{nR}(r,\beta) - j(1/\beta)^{n-3/2} I_{nI}(r,\beta),$$

where

$$I_{nR}(r,\beta) = \int_0^1 \left(\frac{1}{t}\right)^{n+1/2} (1-t^2)^{1/2} J_0(\beta t r) J_{n+3/2}(\beta t)\,dt,$$
$$I_{nI}(r,\beta) = \int_1^{\infty} \left(\frac{1}{t}\right)^{n+1/2} (t^2-1)^{1/2} J_0(\beta t r) J_{n+3/2}(\beta t)\,dt. \tag{29}$$

The first integral in (29) may easily be calculated numerically, whereas the second calls for an alternative procedure. We rewrite I_{nI} accordingly:

$$I_{nI}(r,\beta) = [I_{nI}^{(1)}(r,\beta) + I_{nI}^{(2)}(r,\beta)]/2,$$

where

$$I_{nI}^{(1)}(r,\beta) = \int_1^{\infty} \left(\frac{1}{t}\right)^{n+1/2} \times (t^2-1)^{1/2} J_0(\beta t r) H_{n+3/2}^{(1)}(\beta t)\,dt,$$
$$I_{nI}^{(2)}(r,\beta) = \int_1^{\infty} \left(\frac{1}{t}\right)^{n+1/2} \times (t^2-1)^{1/2} J_0(\beta t r) H_{n+3/2}^{(2)}(\beta t)\,dt. \tag{30}$$

The two right-hand side integrals in (30) may be calculated by means of a contour integration in the complex t plane: the first integral along a contour $\Gamma^{(1)}$ and the second along $\Gamma^{(2)}$. The contours are defined as (see Fig. 1)

$$\begin{aligned}
&\Gamma^{(1)} = t \in (C_r^{(1)} \cup C_R^{(1)} \cup C_i^{(1)} \cup C_c^{(1)}),\\
&C_r^{(1)} = [1,R],\\
&C_R^{(1)} = [Re^{j\varphi} | 0<\varphi<\pi/2],\\
&C_i^{(1)} = [jR,j],\\
&C_c^{(1)} = (e^{j\varphi} | \pi/2>\varphi>0),\\
&R\to\infty, \quad \Gamma^{(2)} \text{ symmetric to } \Gamma^{(1)} \text{ with respect to the real axis.}
\end{aligned} \tag{31}$$

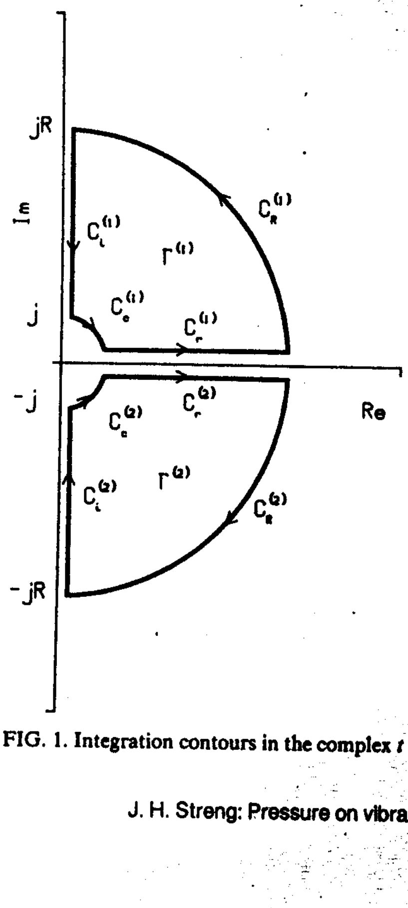

FIG. 1. Integration contours in the complex t plane.

The contributions along $C_R^{(1)}$ and $C_R^{(2)}$ vanish for $R \to \infty$, due to the behavior of $H_{n+3/2}(t)$ for $|t| \to \infty$. For the contribution along $C_i^{(1)}$, where $(t^2-1)^{1/2} = j(1-t^2)^{1/2}$, we have to evaluate

$$\int_{j\infty}^{j} \left(\frac{1}{t}\right)^{n+1/2} j(1-t^2)^{1/2} J_0(\beta tr) H^{(1)}_{n+3/2}(\beta t)dt$$
$$= -\int_1^{\infty} \left(\frac{1}{s}\right)^{n-1/2} (1+s^2)^{1/2} I_0(\beta rs)$$
$$\times (1/j)^{n-3/2} H^{(1)}_{n+3/2}(j\beta s)ds. \qquad (32)$$

Using the relation[7]

$$(1/j)^{n-3/2} H_{n+3/2}(j\beta s) = (-1)^n K_{n+3/2}(\beta s) 2/(j\pi), \qquad (33)$$

it may be shown that the integral in (32) is purely imaginary, whereas the original integral (30) is real valued. Hence, the contribution along $C_i^{(1)}$ is of no importance. The same holds for the contribution along $C_i^{(2)}$, where $(t^2-1)^{1/2} = -j(1-t^2)^{1/2}$. Finally, we calculate the contribution along the quarter-circle segments $C_c^{(1),(2)}$, where we substitute $t = e^{j\varphi}$. Using simple geometry we may show that

$$(t^2-1)^{1/2} = \begin{cases} \overline{Z(|\varphi|)}, & 0 > \varphi > -\pi/2, \\ Z(\varphi), & 0 < \varphi < \pi/2, \end{cases} \qquad (34)$$

where $Z(\varphi) = (2\sin\varphi)^{1/2} e^{j(\varphi+\pi/2)/2}$ and the horizontal bar denotes complex conjugation.

Now, if we substitute (34) into the right-hand side of (30), together with $t = e^{j\varphi}$ and the following identities[7]:

$$J_0(\bar{z}) = \overline{J_0(z)}, \quad H^{(2)}_{n+3/2}(\bar{z}) = \overline{H^{(1)}_{n+3/2}(z)}, \qquad (35)$$

we obtain the contribution along $C_c^{(1)} + C_c^{(2)}$, which can be written as

$$I_{nI}^{(1)+(2)}(r,\beta) = \int_0^{\pi/2} (2\sin\varphi)^{1/2} \mathrm{Im}\,[e^{j[\varphi(1-n)+\pi/4]} J_0(r\beta e^{j\varphi}) \times H^{(1)}_{n+3/2}(\beta e^{j\varphi})]d\varphi. \qquad (36)$$

Finally, we may use the residue statement for the contours $\Gamma^{(1),(2)}$ and return to (29), where we now have

$$I_n(r,\beta) = (1/\beta)^{n-3/2} I_{nR}(r,\beta) + (1/\beta)^{n-3/2} j I_{nI}^{(1)+(2)}(r,\beta). \qquad (37)$$

Both right-hand side integrals in (37) can be calculated numerically without severe problems for nonzero values of β.

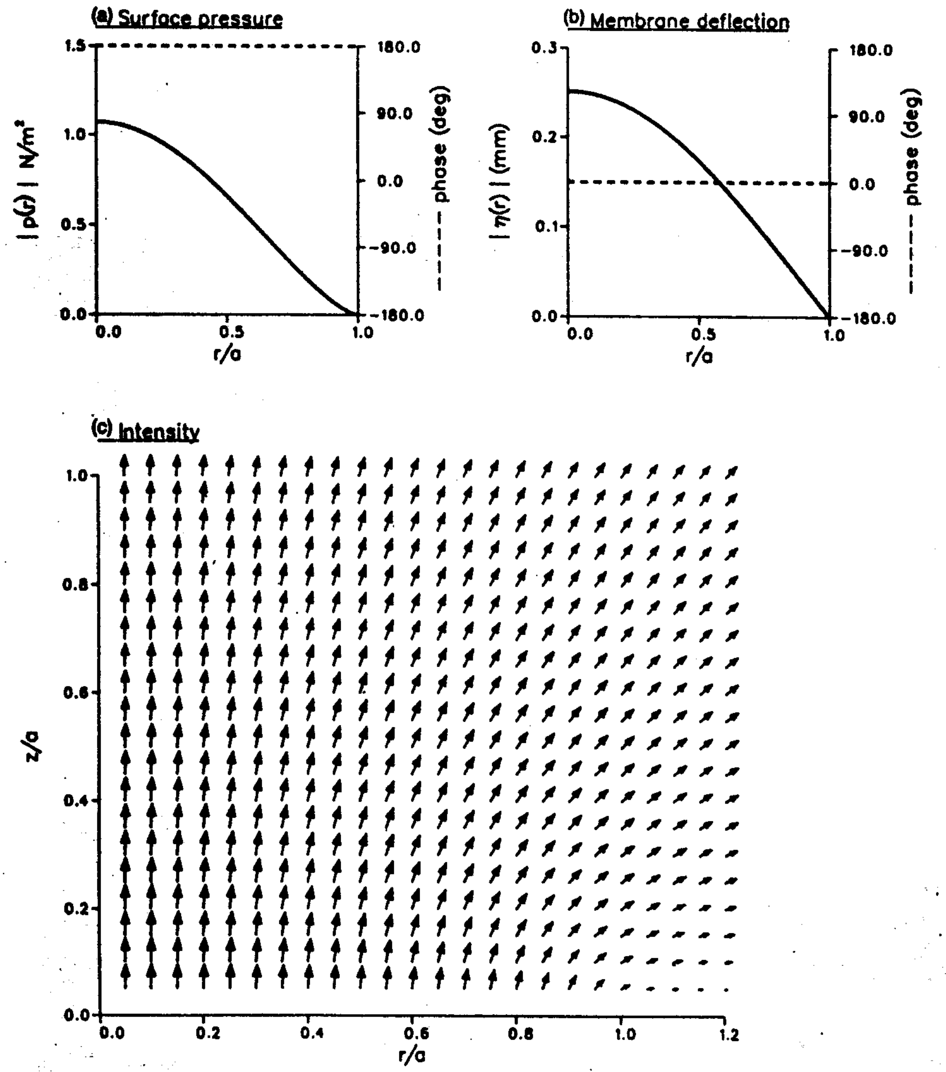

FIG. 2. Vibrating membrane in air at 50 Hz. Membrane parameters: diameter, 0.25 m; tension, 40 N/m; weight, 0.01 kg/m²; driving force amplitude, 1 N/m²; (a) surface pressure: solid = modulus, dashed = phase; (b) membrane deflection: solid = modulus, dashed = phase; and (c) time-averaged intensity: maximum vector length, 0 dB; zero length, < −30 dB.

III. RESULTS

A. The vibrating membrane in air

As an example, we examine the following case:

$$a = 0.125 \text{ m}, \quad T = 40 \text{ N/m}, \quad \rho = 0.01 \text{ kg/m}^2,$$
$$F = 1 \text{ N/m}^2, \quad c_0 = 343 \text{ m/s}, \quad \rho_0 = 1.18 \text{ kg/m}^3, \tag{38}$$

which corresponds to a loudspeakerlike vibrating membrane in air. The collocation points in the calculation of the results of this section were chosen at equal intervals along the radius $0<r<1$. The calculations showed a fast convergence of the power series (12). For low frequencies ($\beta = ka \simeq 0.1$), the first four coefficients were sufficient to give good results, i.e., $a_N < 10^{-3} \max\{a_{N-1},...,a_1\}$, whereas for higher frequencies ($\beta = ka \simeq 0.6$), the number increased to about seven. A maximum relative integration error of 10^{-8} and truncation of the Green's function summation in (26) at $m = 320$ were chosen to obtain the results of this section.

B. Calculated surface pressure, membrane deflection, and intensity

We focus on the lower frequency region at frequencies 50 ($\beta = 0.1145$) (Fig. 2), 150 ($\beta = 0.3435$) (Fig. 3), and 300 Hz ($\beta = 0.6869$) (Fig. 4). The calculated intensity is represented by arrows, pointing in the direction of the intensity. The length of the arrows is a linear measure for the logarithmic intensity; the length of the longest arrow is set to 0 dB; zero length then indicates -30 dB or less.

At 50 Hz we have a masslike reaction of the surrounding air, which can be calculated from the ratio of the surface pressure [Fig. 2(a)] and the membrane deflection [Fig. 2(b)]. The calculated intensity [Fig. 2(c)] shows a typical doubletlike flow of power: maximum radiation in forward direction and no radiation in the plane of symmetry. At 150 Hz (Fig. 3), we get a sharp peak in the acoustic impedance caused by a nodal line in the membrane deflection [Fig. 3(b)]. At the same time, we see that the phase of the surface pressure [Fig. 3(a)] has made a 180° change compared with the 50-Hz case, indicating that we have passed the fundamental resonance frequency of the "mass-spring system" built up by the air mass and membrane tension (a frequency scan shows a resonance frequency of approximately 65 Hz in the case in question). The calculated intensity [Fig. 3(c)] clearly shows the effect of the two membrane regions vibrating in opposite phase: The central part of the membrane radiates power, whereas the outer part is receiving power. At 300 Hz (Fig. 4) the situation is even more dramatic. Here, we have two nodal lines [Fig. 4(b)], and the calculated in-

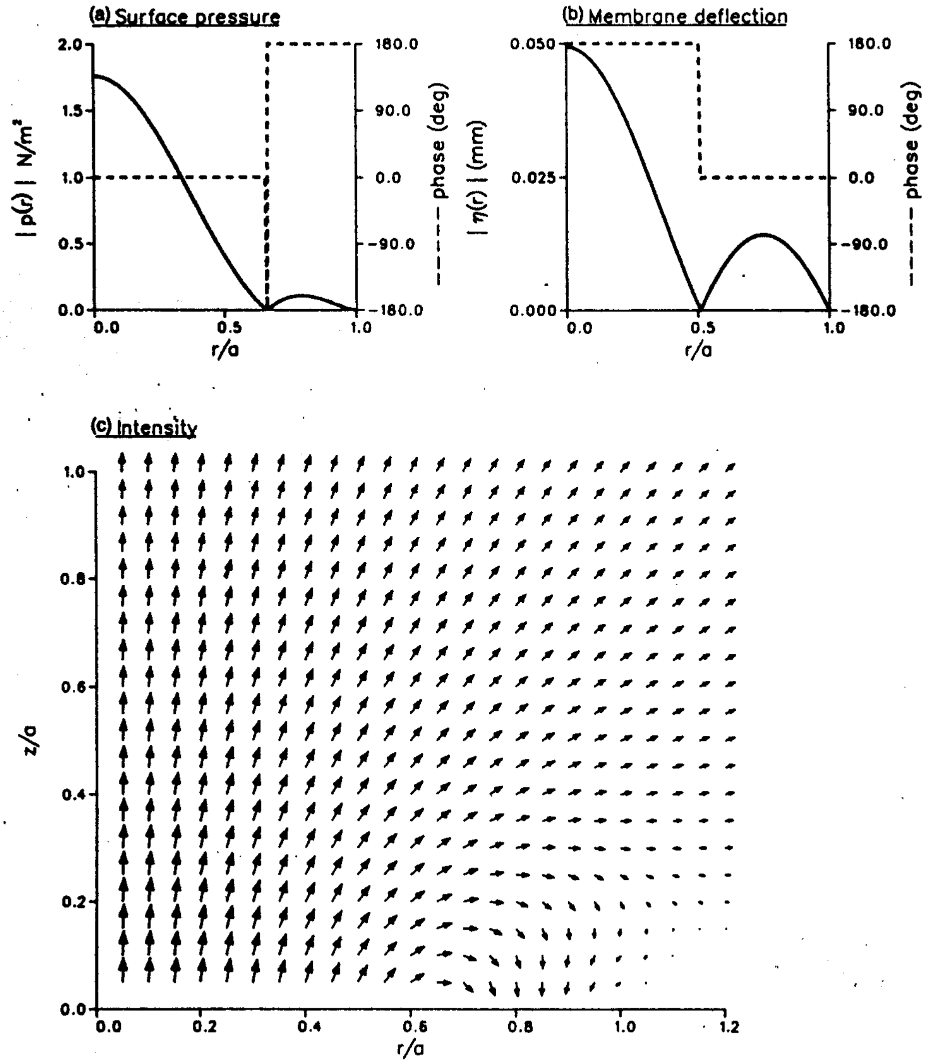

FIG. 3. Vibrating membrane in air at 150 Hz. Membrane parameters: diameter, 0.25 m; tension, 40 N/m; weight, 0.01 kg/m²; driving force amplitude, 1 N/m²; (a) surface pressure: solid = modulus, dashed = phase; (b) membrane deflection: solid = modulus, dashed = phase; and (c) time-averaged intensity: maximum vector length, 0 dB; zero length, < − 30 dB.

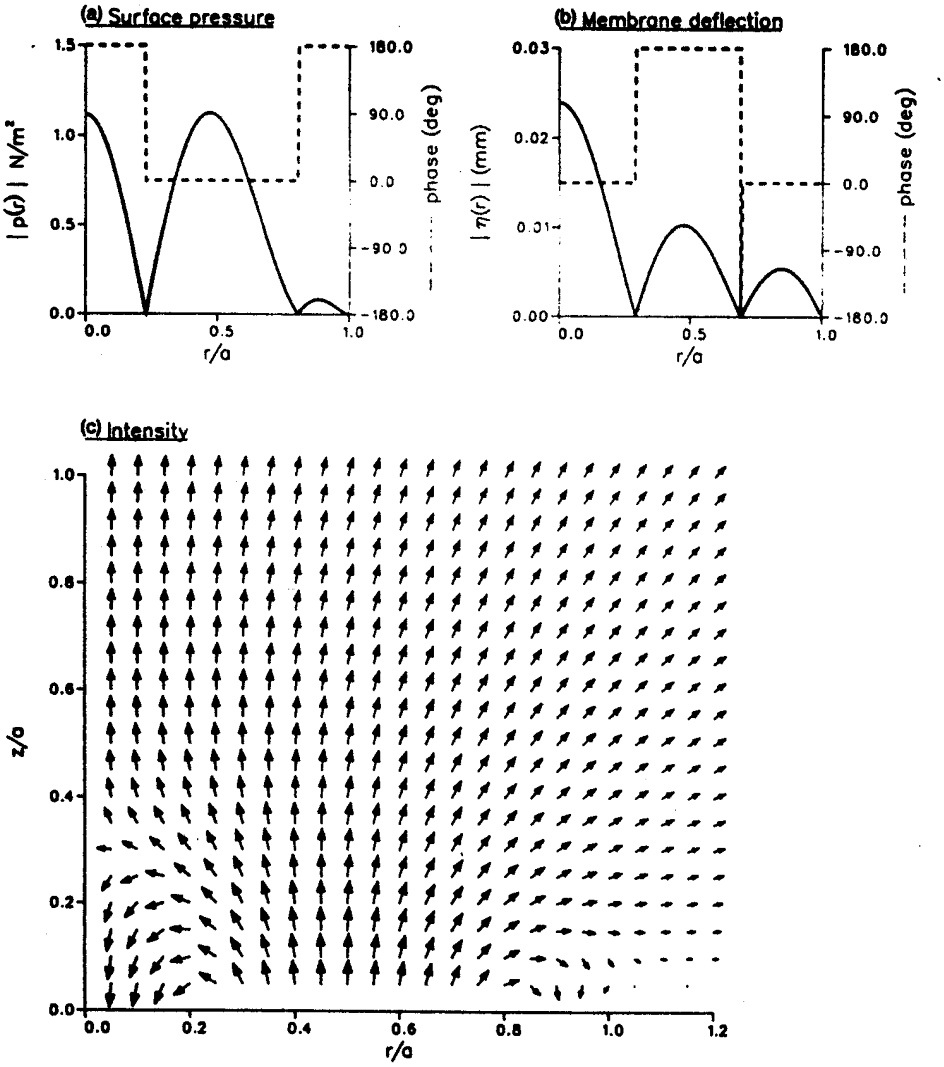

FIG. 4. Vibrating membrane in air at 300 Hz. Membrane parameters: diameter, 0.25 m; tension, 40 N/m; weight, 0.01 kg/m²; driving force amplitude, 1 N/m²; (a) surface pressure: solid = modulus, dashed = phase; (b) membrane deflection: solid = modulus, dashed = phase; and (c) time-averaged intensity: maximum vector length, 0 dB; zero length, < − 30 dB.

tensity [Fig. 4(c)] shows a strong interaction between the various membrane regions through the surrounding air.

The results presented in this section are merely meant to demonstrate the use of the method described in the previous sections. More detailed results are reserved for future articles.

IV. FINAL REMARKS

In the previous section we have calculated the membrane surface pressure by means of collocation, with collocation points at equal distances. This choice is not at all trivial, and it is worthwhile to investigate the existence of an optimum spacing. Another approach is to carry out a least-squares procedure on Eq. (26) in order to determine the coefficients a_n; i.e., we may write (26) as

$$\sum_{n=1}^{\infty} a_n F_n(r) = C(r). \tag{39}$$

We can try to minimize the function

$$E(a_1,...,a_N) = \int_0^1 \left(C(r) - \sum_{n=1}^{N} a_n F_n(r)\right)^2 r\,dr, \tag{40}$$

where a minimum is reached when $\partial E/\partial a_i = 0$. If we carry out this operation, we see that a set of linear equations is obtained from which the coefficients a_n may be calculated:

$$\sum_{n=1}^{N} a_n \int_0^1 F_n(r)F_i(r)r\,dr = \int_0^1 F_i(r)C(r)r\,dr, \quad i = 1,...,N. \tag{41}$$

It must be remarked that Eq. (26), from which the power series coefficients are calculated, must be handled with some care in the numerical evaluation. For large values of n (say $n > 10$), the factor $2^{n+1/2}\Gamma(n+3/2)$ becomes very large, whereas the functions I_n and Ψ_n become very small. This behavior can cause numerical problems, and it may be necessary to carry out extended precision calculations. Calculation of coefficients with large indices will only be necessary, however, for the high-frequency range, so one could also use the high-frequency approximation, Eqs. (20) and (27).

The reader may be interested in the reason why the power series expansion (12) works quite satisfactorily in solving the problem, whereas an expansion of $q_+(r)$ in terms of, for example, the eigenfunctions $J_0(j_{0n}r)$ of the membrane, usually does not. The main reason is, of course, that the expansion (12) fits the fluid behavior because it results directly from the (transformed) Helmholtz equation, and it incorporates *a priori* the correct edge behavior of the surface pressure.

Another point of view may be to regard Eq. (16) as the description of a filtering process; i.e., we rewrite (16) as

$$w(r) = \int_0^\infty (-j\sigma)\left(\int_0^1 q_+(r_0)J_0(\mu r_0)r_0\,dr_0\right)J_0(\mu r)\mu\,d\mu, \tag{42}$$

which states that the (Hankel) transform of $q_+(r)$ is filtered in the μ-domain by a "transfer function" $(-j\sigma)$, whereafter the inverse transform delivers $w(r)$. For high frequencies ($\beta \gg 1$), it may be seen from the definition of σ in (13) that this transfer function is essentially constant over the range where the μ-transform of $q_+(r)$ has a significant value. This "all-pass" effect will result in an approximately equal shape of $w(r)$ and $q_+(r)$ at high frequencies [see result (20)]. For low frequencies ($\beta < 1$ or $\beta \simeq 1$), however, the filter's influence is essential; if we expand $q_+(r)$ at low frequencies in terms of the eigenfunctions $J_0(j_{0n}r)$ of the membrane [which does not *a priori* fulfill the edge constraint $\partial q_+(r)/\partial r|_{r=1} = 0$], then it is easy to show that the "filtered" μ-transform of $q_+(r)$ has important high-μ components, whereas the expected $w(r)$ (smooth at low frequencies) has not. In consequence, we may expect a bad convergence of the chosen power series because somehow we have to cancel these high-μ components. Using Eq. (17), it is easy to show that the filtered μ-transform of expansion (12) does not contain high-μ components and thus may be expected to show a better convergence.

Several other Bessel-like expansions have been investigated and it turns out that bad convergence always corresponds with high-μ components in the filtered μ-transform of the terms of the chosen expansion.

It must be remarked that Bouwkamp[4] prefers an expansion of $q_+(r)$ in terms of (orthogonal) Legendre polynomials, i.e.,

$$q_+(r) = \sum_{n=0}^{\infty} a_n P_{2n+1}[(1-r^2)^{1/2}], \tag{43}$$

which has some advantages in the analysis of the free vibrating disk, but essentially does not differ from (12).

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