$$P(r,z) = \sum_m \left( A_m J_0(k_r \cdot r) e^{-i \cdot k_z \cdot z} \right) \qquad k_z = \sqrt{k^2 - k_r^2} \qquad k_r = \frac{\gamma_m}{a}$$

where the A's are the modal contributions

a = radius of tube

gamma = m constants

note that kr depends on m the modal order.  For m = 0 kr is zero and kz = k

let

$$kz_m = \sqrt{k^2 - \frac{\gamma_m^{\;2}}{a^2}}$$

at each of the n measurements points (r=a) at distance zn from the driven end we will have

$$P_1 = A_0 \cdot e^{-i \cdot k \cdot z_1} + A_1 \cdot J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_1} + A_2 \cdot J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_1}$$

$$P_2 = A_0 \cdot e^{-i \cdot k \cdot z_2} + A_1 \cdot J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_2} + A_2 \cdot J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_2}$$

$$P_3 = A_0 \cdot e^{-i \cdot k \cdot z_3} + A_1 \cdot J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_3} + A_2 \cdot J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_3}$$

we can rewrite this as

$$P = JE \cdot A$$

Where P is the vector of complex velocities and A the vector of complex modal amplitudes (TBD)

and

$$JE = \begin{pmatrix} e^{-i \cdot kz_0 \cdot z_1} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_1} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_1} \\ e^{-i \cdot kz_0 \cdot z_2} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_2} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_2} \\ e^{-i \cdot kz_0 \cdot z_3} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_3} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_3} \end{pmatrix}$$

finally the desired modal amplitudes are simply

$$A = JE^{-1} \cdot P$$

The exension to more data points is trivial, and the extension to a rotated driver is also quite straightforward.

$$\begin{pmatrix} e^{-i \cdot k \cdot z_1} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_1} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_1} \\ e^{-i \cdot k \cdot z_2} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_2} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_2} \\ e^{-i \cdot k \cdot z_3} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_3} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_3} \end{pmatrix}^{-1}$$

$$A(k) = \begin{pmatrix} e^{-i \cdot k \cdot z_1} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_1} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_1} \\ e^{-i \cdot k \cdot z_2} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_2} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_2} \\ e^{-i \cdot k \cdot z_3} & J_0(\gamma_1) \cdot e^{-i \cdot kz_1 \cdot z_3} & J_0(\gamma_2) \cdot e^{-i \cdot kz_2 \cdot z_3} \end{pmatrix}^{-1} \cdot P(k)$$