Transfer functions

The transfer of a filter can be described by a transfer function consisting of the ratio between two polynomials in *s*, which in older literature is denoted as *p*. (This actually only applies when the filter is linear, time invariant, continuous time and lumped, but analogue filters used for audio are usually close enough to being all of that to model them with a linear, time-invariant, continuous-time lumped network model). Personally I think the old notation *p* is much clearer than *s*, because *s* is too similar to s, the SI symbol for second. Still, since *s* is the more usual notation, I will stick to it.

Depending on the type of calculation one wants to do, *s* can be regarded as the Laplace variable (outside the scope of this article), as a differentiation to time operator, or as $j\omega$, where ω is the radian frequency ($\omega=2\pi f$) and *j* is the imaginary unit number ($j^2=-1$).

For example, take this transfer function:

$$H(s)=\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}=K\frac{a_2 s^2+a_1 s+1}{b_2 s^2+b_1 s+1}$$

Interpreting *s* as a differentiation-to-time operator, this means that the relation between the input and output voltage is given by this differential equation:

$$b_2\frac{d^2 v_{\mathrm{out}}}{dt^2}+b_1\frac{dv_{out}}{dt}+v_{\mathrm{out}}=K\left(a_2\frac{d^2 v_{\mathrm{in}}}{dt^2}+a_1\frac{dv_{in}}{dt}+v_{\mathrm{in}}\right)$$

Using complex numbers, calculating the output signal becomes relatively simple when the filter does not oscillate and when the input signal is a stationary sine or cosine wave (that is, a sine or cosine that has been there long enough for initial transients to damp out). A cosine equals the sum of two complex exponential functions:

$$\cos(\omega t)=\frac{1}{2}e^{j\omega t}+\frac{1}{2}e^{-j\omega t}$$

As the network model is linear, we may use the superposition principle. That is, we may calculate the response to each complex exponential signal independently and simply add the results.

The time derivative of a complex exponential signal is:

$$\frac{de^{j\omega t}}{dt}=j\omega e^{j\omega t}$$

Hence, when

$$v_{\mathrm{in}}=\frac{1}{2}e^{j\omega t}$$

we get

$$b_2 \frac{d^2 v_{\text{out}}}{dt^2} + b_1 \frac{dv_{\text{out}}}{dt} + v_{\text{out}} = K\left(a_2 (j\omega)^2 + a_1 j\omega + 1\right)\frac{1}{2} e^{j\omega t}$$

This equation is satisfied when the output signal is also a complex exponential signal of the same frequency, but multiplied with some complex multiplication factor. That is, assume that

$$v_{\text{out}} = Xe^{j\omega t}$$

This results in

$$\left(b_2 (j\omega)^2 + b_1 j\omega + 1\right)Xe^{j\omega t} = K\left(a_2 (j\omega)^2 + a_1 j\omega + 1\right)\frac{1}{2} e^{j\omega t}$$

$$X = \frac{1}{2} \bullet \frac{K\left(a_2 (j\omega)^2 + a_1 j\omega + 1\right)}{b_2 (j\omega)^2 + b_1 j\omega + 1} = \frac{1}{2} H(j\omega)$$

and

$$\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{X}{\frac{1}{2}} = H(j\omega)$$

So when you substitute $s=j\omega$, the transfer function turns into a complex-valued gain factor for complex exponential input signals. Note that $H(j\omega)$ can also be written in a polar form:

$$H(j\omega) = |H(j\omega)| e^{j\phi}$$

with

$$\phi = \arctan\left(\text{Im}(H(j\omega)) / \text{Re}(H(j\omega))\right) + k\pi$$

when the real part of $H(j\omega)$ is not zero and where k is an integer. The factor $|H(j\omega)|$ represents the actual gain, while the factor $e^{j\varphi}$ just gives a phase shift of φ.

In the end, we are interested in the response to the real-valued cosine wave rather than to the complex exponential waveform. Hence, when

$$v_{\text{in}} = \cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

then

$$v_{\text{out}} = \frac{1}{2} e^{j\omega t} H(j\omega) + \frac{1}{2} e^{-j\omega t} H(-j\omega)$$

It can be shown that $H(j\omega)$ and $H(-j\omega)$ must have equal real parts and opposite imaginary parts for any filter that produces a real-valued output signal for each real-valued input signal. This results in equal magnitudes, but opposite phases for $H(j\omega)$ and $H(-j\omega)$. Hence,

$$v_{\text{out}} = \frac{1}{2} e^{j\omega t} |H(j\omega)| e^{j\phi} + \frac{1}{2} e^{-j\omega t} |H(j\omega)| e^{-j\phi} = |H(j\omega)| \frac{1}{2}\left(e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}\right) = |H(j\omega)| \cos\left(\omega t + \phi\right)$$

That is, also for a normal cosine wave $|H(j\omega)|$ represents the gain, while the phase shift is φ.

Poles and zeros

The values of *s* for which the denominator of the transfer function is zero are called the poles of the transfer function. The values of *s* for which the numerator of the transfer function is zero are called the zeros. The number of poles of a lumped system can never exceed the number of energy-storing parts (such as capacities and inductivities in an electric network, or masses and springs in a mechanic system).

An interesting property is that all transfers of a system with multiple in- or outputs have the same poles, although in some of these transfers some poles may be covered by zeros. This is a very useful property for loudspeaker design, as it allows one to partly determine the electroacoustic transfer by simple impedance measurements.