

Optimal Use of Some Classical Approximations in Filter Design

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Abstract—The classical Butterworth, Chebyshev and Elliptic (Cauer) low-pass filter approximations can be used in the design of analog and IIR digital filters in such a way as to obtain passband, stopband and transition band optimized filters at no order cost. The exact analytical relationships for such an optimal deployment of these approximations are developed and presented in this paper and their use is demonstrated through design examples.

Index Terms—Butterworth, Cauer, Chebyshev, elliptic filters, equiripple filters, filter approximations, filter theory, filters.

I. INTRODUCTION

THE normalized low-pass filter magnitude specifications are given in terms of plain gain or logarithmic gain in decibels as shown in Fig. 1. H_o is a reference level usually taken equal to 1 (i.e., $a_o = 0$ dB) without loss of generality and therefore filter requirements are determined by three numbers

$$\{\Omega_{so}, H_{co}, H_{so}\} \text{ or } \{\Omega_{so}, a_{\max}, a_{\min}\}$$

where $\Omega_{so} = \frac{\omega_S}{\omega_C}$, $a_{\max} = 20 \log \left(\frac{H_o}{H_{co}} \right)$, $a_{\min} = 20 \log \left(\frac{H_o}{H_{so}} \right)$.

The gain specifications of the filter are approximated in the Butterworth, Chebyshev and Elliptic cases using an approximating function $P(N, \Omega)$ in a gain expression of the form

$$G(N, \Omega) = \frac{H_o}{\sqrt{1 + \varepsilon^2 P^2(N, \Omega)}}. \quad (1)$$

The function $P(N, \Omega)$ is well defined in all cases

$$\text{Butterworth : } P(N, \Omega) = \Omega^N \quad (2a)$$

$$\text{Chebyshev : } P(N, \Omega) = C(N, \Omega) \quad (2b)$$

$$\text{Elliptic : } P(N, \Omega) = R(N, \Omega_s, \Omega) \quad (2c)$$

with $\Omega_s = \Omega_{so}$.

$C(N, \Omega)$ is the Chebyshev polynomial of order N and $R(N, \Omega_s, \Omega)$ is the elliptic (or Chebyshev) rational function defined as

$$R(N, \Omega_s, \Omega) = A \cdot \Omega^\lambda \cdot \prod_{m=1}^{(N-\lambda)/2} \left(\frac{\Omega^2 - \Omega_{Z(m)}^2}{\Omega^2 - \Omega_s^2} \right)$$

$$\text{with } R(N, \Omega_s, 1) = 1 \quad \lambda = \begin{cases} 1, & \text{for odd } N \\ 0, & \text{for even } N \end{cases}. \quad (3a)$$

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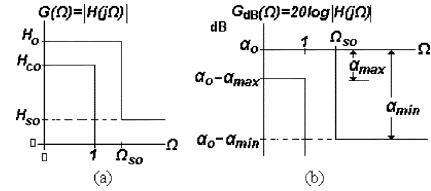


Fig. 1. Normalized LP filter specifications. (a) Plain gain. (b) Logarithmic gain in decibels

The $(N - \lambda)/2$ critical frequencies $\Omega_Z(m)$ are given [1] by

$$\Omega_Z(m) = sn \left[\left(\frac{2m-1}{N} + 1 \right) K \left(\frac{1}{\Omega_s} \right), \frac{1}{\Omega_s} \right]. \quad (3b)$$

It is clear that only in the elliptic case the approximating function P depends on the stopband edge frequency. The complete elliptic integral $K(x)$ having a modulus x and the Jacobi elliptic sine of u with modulus x , $sn(u, x)$, (see Appendix) are responsible for the complexity of the mathematics of the elliptic filters.

In all cases, P is such that $|P(N, 1)| = 1$ and the value $P(N, \Omega_s) = L(N, \Omega_s)$, called the *discrimination factor*, is given by

$$\text{Butterworth : } L(N, \Omega_s) = \Omega_s^N \quad (4a)$$

$$\text{Chebyshev : } L(N, \Omega_s) = \cosh(N \cosh^{-1}(\Omega_s)) \quad (4b)$$

$$\text{Elliptic : } L(N, \Omega_s) = R(N, \Omega_{so}, \Omega_s). \quad (4c)$$

In the elliptic case, $L(N, \Omega_s)$ can be calculated [2] from

$$L(N, \Omega_s) = \frac{\Omega_s^N}{\prod_{m=1}^{(N-\lambda)/2} sn^4 \left[\left(\frac{2m-1+\lambda}{N} + 1 \right) K \left(\frac{1}{\Omega_s} \right), \frac{1}{\Omega_s} \right]}$$

$$\lambda = \begin{cases} 1, & \text{for odd } N \\ 0, & \text{for even } N. \end{cases} \quad (5)$$

Given the specifications $\{\Omega_{so}, H_{co}, H_{so}\}$, the order equation $\varphi(\Omega_{so}, H_{co}, H_{so})$ for each approximation family gives, in general, a non-integer value N_{dec} which is rounded up to the next integer value N_o . Traditionally, the filter of order N_o is then designed using $\varepsilon = \sqrt{(H_o/H_{co})^2 - 1}$, a choice that ensures exact satisfaction of the passband requirement H_{co} and improved stopband performance with $G(N_o, \Omega_{so}) = H_{sa} < H_{so}$, as a trade-off of the increase of the order from N_{dec} to N_o . This “traditional” design is shown in Fig. 2(I) and is referred to as *stopband edge gain optimized*.

The difference $N_o - N_{\text{dec}}$ allows for trade-offs in requirements so that the filter can be optimized for the given order N_o [3]. For example, the specifications can also be satisfied in a manner shown in Fig. 2(II) with minimum passband tolerance H_o to $H_{c\max}$, a value that depends on Ω_{so} and H_{so} . Furthermore, in the elliptic case, due to the dependence of the

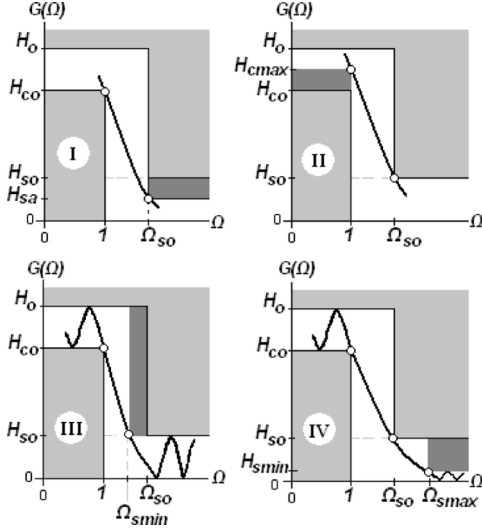


Fig. 2. The optimization cases (I) Stopband edge gain (II) Passband tolerance (III) Transition band (IV) Stopband attenuation.

approximating function P on the stopband edge frequency Ω_{so} , two more optimizations are available. The *transition band* and the *stopband attenuation* optimizations, shown in Fig. 2 (III) and (IV), respectively.

In the transition band optimization of Fig. 2(III), a new stopband edge frequency $\Omega_{smin} < \Omega_{so}$, is calculated from N_o and H_{so} , that leads to stopband gain tolerance equal to the specified H_{so} . In the stopband attenuation optimization of Fig. 2(IV), a new stopband edge frequency $\Omega_{smax} > \Omega_{so}$, can be calculated from N_o and H_{so} , that leads to stopband gain tolerance equal to the specified H_{so} (i.e., the gain at $\Omega_{so} = H_{so}$) but with much higher attenuation (lower gain) for $\Omega_{so} > \Omega_{smax}$.

In all cases, when the optimized requirements are used in the order equation, the result will be exactly the integer N_o , i.e.,

$$\begin{aligned} \varphi(\Omega_{so}, H_{co}, H_{sa}) &= N_o \quad \varphi(\Omega_{so}, H_{cmax}, H_{so}) = N_o \\ \varphi(\Omega_{smin}, H_{co}, H_{so}) &= N_o \quad \varphi(\Omega_{smax}, H_{co}, H_{smin}) = N_o. \end{aligned}$$

In [3], a method is developed using nomographs and the band edge selectivity (BES) concept to calculate approximately H_{cmax} , Ω_{smin} , Ω_{smax} and H_{smin} for the optimized designs. In this paper the exact equations are presented for the most common types of approximation and the results of an example design are compared to those presented in [3].

II. DESIGN EQUATIONS AND OPTIMIZATION

Given $\{\Omega_s, H_c, H_s\}$, the integer order N_o can be calculated from the order equation of each approximation case. If we require full exploitation of both passband and stopband tolerances for given integer order N_o we must satisfy the following two design equations:

$$G(N_o, 1) = \frac{H_o}{\sqrt{1 + \varepsilon^2}} = H_c \quad (6a)$$

$$G(N, \Omega_s) = \frac{H_o}{\sqrt{1 + \varepsilon^2 L^2(N_o, \Omega_s)}} = H_s \quad (6b)$$

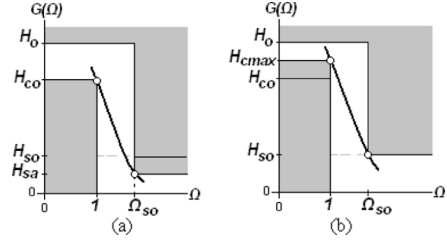


Fig. 3. Optimization for: (a) stopband edge gain and (b) passband gain.

or equivalently

$$\varepsilon = \sqrt{\frac{H_o^2}{H_c^2} - 1} \quad (7)$$

$$\frac{\sqrt{\left(\frac{H_o^2}{H_s^2} - 1\right)}}{L(N_o, \Omega_s)} = \sqrt{\left(\frac{H_o^2}{H_c^2} - 1\right)} (= \varepsilon). \quad (8)$$

For given H_o and N_o , (8) relates the three filter requirements $\{\Omega_s, H_c, H_s\}$ that can be exactly realized with order N_o . This means that if we specify two of them the third can be calculated solving (8).

A. Stopband Edge Gain Optimized Filter

In this case, $\Omega_s = \Omega_{so}$ and $H_c = H_{co}$ are specified and solution of (8) for H_s gives the minimum value H_{sa} required with order N_o

$$\begin{aligned} \varepsilon &= \sqrt{\frac{H_o^2}{H_{co}^2} - 1} \\ H_s &= \frac{H_o}{\sqrt{1 + \left(\frac{H_o^2}{H_{co}^2} - 1\right) L^2(N_o, \Omega_{so})}} = H_{sa}. \end{aligned} \quad (9)$$

Design of the filter with $\{\Omega_{so}, H_{co}, H_{sa}\}$ leads to the stopband edge gain optimized filter of order N_o [Fig. 3(a)]. This in fact is the standard procedure leading to a filter designed from $\{\Omega_{so}, H_{co}, H_{so}\}$ which actually realizes $\{\Omega_{so}, H_{co}, H_{sa}\}$.

B. Passband Tolerance Optimized Filter

In this case, $\Omega_s = \Omega_{so}$ and $H_s = H_{so}$ are specified and solution of (8) for H_c gives the maximum value H_{cmax} that can be required with order N_o .

$$H_{cmax} = \frac{H_o}{\sqrt{1 + \frac{\frac{H_o^2}{H_{so}^2} - 1}{L^2(N_o, \Omega_{so})}}}; \quad \varepsilon = \sqrt{\frac{H_o^2}{H_{cmax}^2} - 1}. \quad (10)$$

Design of the filter with $\{\Omega_{so}, H_{cmax}, H_{so}\}$ leads to the *passband tolerance optimized* filter of order N_o [Fig. 3(b)]. The pole positions of the transfer function are different from those of the stopband edge gain optimized filter. It should be noticed however that in the elliptic case the transmission zeros do not change since the elliptic rational function depends only on the order and the stopband edge frequency, parameters that are common in both cases.

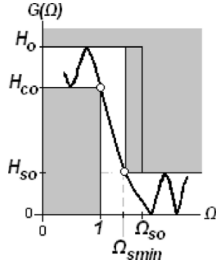


Fig. 4. Elliptic transition band optimized filter.

C. Transition Band Optimized Filter

If $H_c = H_{co}$ and $H_s = H_{so}$ are specified, solution of (8) for Ω_s gives the minimum value Ω_{smin} of the stopband edge frequency that can be used with order N_o to exactly satisfy the gain requirements with $G(1) = H_{co}$ and $G(\Omega_{smin}) = H_{so}$. In this case, only $L(N_o, \Omega_s)$ can be directly calculated from (8)

$$L(N_o, \Omega_s) = \frac{1}{g} \quad (11)$$

where $\varepsilon = \left(\frac{H_o^2}{H_c^2}\right) - 1$ and $g = \frac{\varepsilon}{\sqrt{\left(\left(\frac{H_o^2}{H_s^2}\right) - 1\right)}}$.

In fact (11) gives the relationship of Ω_s and the gain requirements which can be exactly satisfied with N_o .

Calculation of Ω_s from $L(N_o, \Omega_s)$ is straightforward for the Butterworth and Chebyshev cases using (4a) and (4b) respectively. Since Ω_s is the minimum value of the stopband edge frequency that satisfies exactly the gain requirements, is denoted as Ω_{smin}

$$\text{Butterworth : } \Omega_{smin} = g^{-1/N_o} \quad (12a)$$

$$\text{Chebyshev : } \Omega_{smin} = \cosh\left(\frac{1}{N_o} \cosh^{-1}\left(\frac{1}{g}\right)\right). \quad (12b)$$

For the elliptic case, Huber in [2] has shown that since the relation of g and Ω_s (when $G(\Omega_s) = H_s$), is given by (11) as $g = 1/L(N_o, \Omega_s)$, Ω_s is given in terms of g by

$$\Omega_s = \frac{1}{\sqrt{1 - \frac{1}{L^2\left(N_o, \frac{1}{\sqrt{1-g^2}}\right)}}} \doteq \Omega_{smin}. \quad (13)$$

Since the elliptic rational function $R(N_o, \Omega_s, \Omega)$ depends on Ω_s and so do the poles and zeros of the transfer function [4], requiring Ω_{smin} in the specifications instead of Ω_{so} . This leads to a completely different filter of the same order N_o that exactly satisfies the stopband and passband specifications H_{co} and H_{so} . This filter is referred to as the *transition band optimized* filter [Figs. 2 (III) and 4].

In the Butterworth and Chebyshev cases, the poles of the transfer function do not depend on the stopband edge frequency Ω_s [4], [5]. Thus, requiring Ω_{smin} instead of Ω_{so} in the design procedure does not produce a different filter as in the elliptic case, thereby making Ω_{smin} simply the frequency at which $G(N_o, \Omega_{smin}) = H_{so}$.

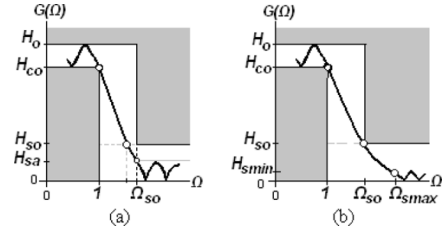


Fig. 5. (a) Typical. (b) Stopband attenuation optimized.

D. Stopband Attenuation Optimized Filter

In the elliptic case more space for optimization exists due to the dependence of the approximating elliptic rational function $R(N, \Omega_s, \Omega)$ on the stopband edge frequency and a fourth design is therefore of special interest.

Fig. 5(a) shows the gain response of an elliptic filter designed with $\{\Omega_{so}, H_{co}, H_{so}\}$. According to (9), the gain at the stopband edge frequency is $H_{sa} < H_{so}$. It is obvious that increasing the value of the stopband edge frequency, a value $\Omega_s = \Omega_{smax}$ exists at which $G(N_o, \Omega_{so}) = H_{so}$ as shown in Fig. 5(b).

Referring to Fig. 5, Ω_{smax} and H_{smin} are related with (11) as follows:

$$\frac{\left(\frac{H_o^2}{H_{co}^2} - 1\right)}{\left(\frac{H_o^2}{H_{smin}^2} - 1\right)} = \frac{1}{L^2(N_o, \Omega_{smax})}. \quad (14a)$$

From which we get

$$H_{smin} = \frac{H_o}{\sqrt{1 + \left(\frac{H_o^2}{H_{co}^2} - 1\right) L^2(N_o, \Omega_{smax})}}. \quad (14b)$$

The following equation yields Ω_{smax} :

$$\frac{H_o}{\sqrt{1 + \varepsilon^2 R^2(N_o, \Omega_{smax}, \Omega_{so})}} = H_{so} \quad (15a)$$

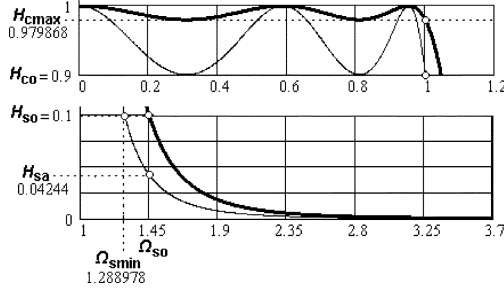
or equivalently equation

$$R^2(N_o, \Omega_{smax}, \Omega_{so}) = \frac{1}{g^2} \quad (15b)$$

with $g = \sqrt{\frac{\left(\left(\frac{H_o^2}{H_{co}^2}\right) - 1\right)}{\left(\left(\frac{H_o^2}{H_{so}^2}\right) - 1\right)}}$.

This means that the elliptic rational function $R(N_o, \Omega_{smax}, \Omega)$ must be determined for the unknown stopband edge frequency Ω_{smax} with the additional condition that its value for $\Omega = \Omega_{so}$ must be $\pm 1/g$. This could not be done analytically, given the inherent difficulty in determining $R(N_o, \Omega_s, \Omega)$ even for known $\{N_o, \Omega_s\}$ and an algorithmic approach is inevitable. This is embodied in the software in [6]. The algorithms calculate the elliptic rational function for N_o and any stopband edge frequency Ω_{sx} . The gain with specified H_{co}

$$G(N_o, \Omega_{so}) = \frac{H_o}{\sqrt{1 + \varepsilon^2 R^2(N_o, \Omega_{sx}, \Omega_{so})}}$$


 Fig. 6. Chebyshev example ($N_o = 5$).

is calculated for stopband edge frequencies from $\Omega_{sx} = \Omega_{so}$ upwards, until it becomes equal to the specified H_{so} .

III. DESIGN EXAMPLES

For a Chebyshev filter with specifications $H_o = 1$, $\Omega_{so} = 1.45$, $H_{co} = 0.9$ and $H_{so} = 0.1$, the order was found $N_o = 5$. From (4b) we find $L(5, 1.45) = 48.833245$ and from (9), (10) and (12b) respectively, $H_{sa} = 0.04244$, $H_{cmax} = 0.979868$ and $\Omega_{smin} = 1.288978$. The transfer function [5] of the typical and the passband tolerance optimized designs are

$$H_a(s) = \frac{0.129046}{(s + \omega_R) \left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2 \right) \left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2 \right)}$$

with $\omega_R = 0.2987$, $\omega_1 = 0.6593$, $Q_1 = 1.3641$, $\omega_2 = 0.9969$, $Q_2 = 5.3996$ and

$$H_b(s) = \frac{0.306743}{(s + \omega_R) \left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2 \right) \left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2 \right)}$$

with $\omega_R = 0.4751$, $\omega_1 = 0.7558$, $Q_1 = 0.9831$, $\omega_2 = 1.0631$, $Q_2 = 3.6206$.

Fig. 6 shows the responses of the typical design (stopband edge gain optimized filter) and of the passband tolerance optimized filter. As mentioned, in this case Ω_{smin} is simply the frequency at which the gain becomes equal to H_{so} .

As a second example, an elliptic filter transfer function with specifications

$$\Omega_{so} = 2 \quad H_o = 1 \quad H_{co} = 0.86596432 \quad H_{so} = 0.01$$

$$(\Omega_{so} = 2 \quad a_o = 0 \quad a_{max} = 1.25 \text{ dB} \quad a_{min} = 40 \text{ dB})$$

will be designed and optimized. The order of the filter is calculated from (19): $N_{dec} = 3.2548 \rightarrow N_o = 4$.

The transfer function of the fourth-order elliptic filter will be of the form

$$H(s) = \frac{C (s^2 + \omega_{1z}^2) (s^2 + \omega_{2z}^2)}{\left(s^2 + \frac{\omega_{1p}}{Q_1}s + \omega_{1p}^2 \right) \left(s^2 + \frac{\omega_{2p}}{Q_2}s + \omega_{2p}^2 \right)}$$

and parameters C , ω_{1z} , ω_{2z} , ω_{1p} , $Q_1\omega_{2p}$, and Q_2 are calculated for all for optimization cases.

A. Stopband Edge Gain Optimized Filter

From (5) we find $L(4, 2) = 773.9781$. From (9) we get $H_{sa} = 0.002237$ or 53.0058 dB. The corresponding transfer function can be calculated [6], where $C = 0.0223722$, $\omega_{1z} = 2.143189$, $\omega_{2z} = 4.922113$, $\omega_{1p} = 0.543167$, $Q_1 = 0.841554$, $\omega_{2p} = 0.98715$, $Q_2 = 4.40412$.

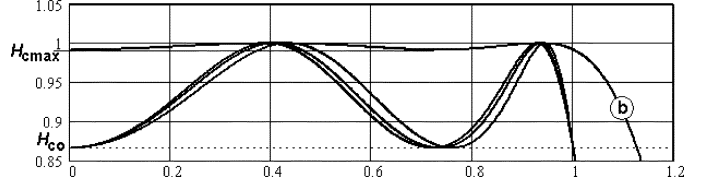


Fig. 7. Passband responses of the example design.

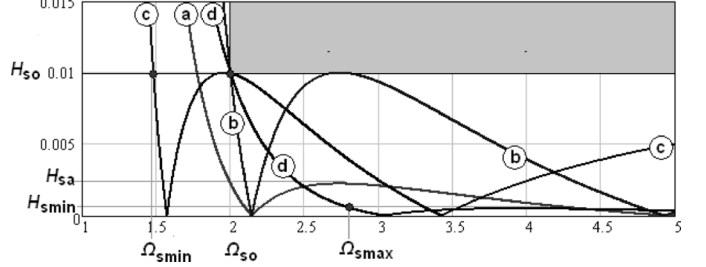


Fig. 8. Stopband responses of the example design.

B. Passband Tolerance Optimized Filter

The discrimination factor is again $L(4, 2) = 773.9781$. From (10) we find $H_{cmax} = 0.991757$ or 0.0719 dB (≈ 0.1 dB in [3]). The parameters of the transfer function in this case will be $C = 0.01$, $\omega_{1z} = 2.143189$, $\omega_{2z} = 4.922113$, $\omega_{1p} = 0.910048$, $Q_1 = 0.631204$, $\omega_{2p} = 1.163993$, $Q_2 = 2.540007$.

C. Transition Band Optimized Filter

From (13) in combination with (5) we find $\Omega_{smin} = 1.480573$ ($= 1.5$ in [3]). From (5), $L(4, 1.480573) = 173.1473$. The parameters of the transfer function in this case are $C = 0.01$, $\omega_{1z} = 1.57073$, $\omega_{2z} = 3.419262$, $\omega_{1p} = 0.582288$, $Q_1 = 0.866334$, $\omega_{2p} = 0.991177$, $Q_2 = 5.179597$.

D. Stopband Attenuation Optimized Filter

Using our algorithm as embodied in [6] we find $\Omega_{smax} = 2.81783$ (≈ 2.5 in [3]). Using (5) we find $L(4, 2.81783) = 3532.7836$ and from (14a) we find $H_{smin} = 0.00049$ or 66.1936 dB (≈ 58 dB in [1]). The parameters of the transfer function are $C = 0.00049014$, $\omega_{1z} = 3.035342$, $\omega_{2z} = 7.157089$, $\omega_{1p} = 0.524748$, $Q_1 = 0.830623$, $\omega_{2p} = 0.984925$, $Q_2 = 4.086599$.

Figs. 7 and 8 show the passband and stopband details of the four designs.

IV. CONCLUSION

The classical approximations are traditionally used in analog and IIR digital filter design in such a way as to satisfy the specified passband tolerance in an exact way with minimum gain at the stopband edge frequency [Fig. 2(I)]. It is shown in this paper that filters with Butterworth, Chebyshev and elliptic response can be designed with optimum passband behaviour [i.e., lower gain tolerance, Fig. 2 (II)] at no order cost and that the dependence of the approximating elliptic rational function, on the stopband edge frequency, allows two more designs: one with optimized transition band [Fig. 2(III)] and one with optimized stopband attenuation [Fig. 2(IV)]. In contrast to other relevant

