An Easier Implementation of a 4^{th} order State-Variable Active Filter

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Contents

1 Introduction

This document describes the dnalysis and esign of a 4^{th} order State Variable filter with either a Linkwitz-Riley transfer characteristic or a Butterworth characteristic.

A State Variable filter is a very elegant solution if a filter with both high-pass and low-pass outputs at the same cut-off frequency are required, like in audio active cross-over applications. Because both frequencies are equal by design, component selection or frequency adjustment becomes quite easy. Deviations of 5-10% of a cross-over frequency do not usually have any effects on performance as an audio filter.

The contents relies heavily on a technical paper written by Janne Ahonen: "The analysis of fourth-order state variable filter and it's application to Linkwitz-Riley filters" [1]. However, the circuit design described in that paper is somewhat complicated due to interdepecies of components in the feedback loop.

The analysis as described here uses a different approach with an active inverter element in the feedback loop. This allows for very easy adjusting the gain and Q of the filter. In addition it is demonstrated that the LR characteristic can easily be converted to a 4^{th} order Butterworth filter.

This design does not offer any better performance than Ahonen's design. It is also questionable if the more complicated synthesis actually is such a burden given the todays available calculation applications. For me it was just fun to see if it could be done.

I should emphasize that without the paper of Ahonen, I would not have been able to understand the analysis of the 4^{th} order State Variable filter, and hence I could not have changed the design to something easier to implement.

2 4^{th} order Linkwitz-Riley compared to Butterworth

An electronic filter circuit is broadly characterized by its characteristic or Q, cut-off frequency ω_0, its category (low-pass, high pass, etc) and its order. Common for all filters is that *by definition* the attenuation is $-3dB$ at the cut-off frequency.

This also holds for a 4^{th} order filter other than Linkwitz-Riley. Despite of the attenuation of $24db/oct$ the attenuation at the defined ω_0 still is $3dB$.

However, the 4^{th} order Linkwitz-Riley filter has a defined attenuation of $6dB$ at ω_0. This is preferable for audio applications [1] and enables simple cascading of two 2^{nd} order Butterworth filters with an equal Q of 0.707 each, yielding an attenuation of $6dB$ at ω_0.

The design of the 4^{th} order State Variable filter in this report decribes both options, a 4^{th} order Butterworth with $3dB$ attenuation at ω_0 and a 4^{th} order Linkwitz-Riley with $6dB$ attenuation at ω_0.

[1] More specific for loudspeaker active crossover filters. Since the resulting sound pressure level of two speakers driven in phase is summed, the attenuation for each driver should be $6dB$ at the crossover frequency.

The 4^{th} order Butterworth is formed by changing the Q's of each 2^{nd} order transfer functions to an appropriate value.

Since the Linkwithz-Riley filter is considered optimal for analog audio applications, the 4^{th} order Butterworth is not intended as an alternative, but it might be applicable in other applications.

3 4^{th} order transfer function

Note: Equations 1 thru 4 are copied from [1]

The transfer function $H(s)$ for a 2^{nd} order high pass filter is:

$$H_{hp2}(s) = \left(\frac{Ks^2}{s^2 + s\left(\frac{\omega_0}{Q}\right) + \omega_0^2} \right) \tag{1}$$

A 4^{th} order filter can be synthesized by cascading two 2^{nd} order filters For the transfer function is obtained by multiplying two 2^{nd} order functions:

$$H_{hp4}(s) = H_{hp2}(s) \cdot H_{hp2}(s) = \left(\frac{K_1 s^2}{s^2 + s\left(\frac{\omega_{01}}{Q_1}\right) + \omega_{01}^2} \right) \cdot \left(\frac{K_2 s^2}{s^2 + s\left(\frac{\omega_{02}}{Q_2}\right) + \omega_{02}^2} \right) \tag{2}$$

$$H_{hp4}(s) = \frac{K_1 K_2 s^4}{s^4 + s^3\left(\frac{\omega_{01}}{Q_1} + \frac{\omega_{02}}{Q_2}\right) + s^2\left(\omega_{01}^2 + \frac{\omega_{01}\omega_{02}}{Q_1 Q_2} + \omega_{02}^2\right) + s\left(\frac{\omega_{01}^2\omega_{02}}{Q_1} + \frac{\omega_{01}\omega_{02}^2}{Q_2}\right) + \omega_{01}^2\omega_{02}^2} \tag{3}$$

This transfer function can be simplified by these assumptions:

$\omega_{01} = \omega_{02} = \omega_0$

$K_1 = K_2 = K$

but deviating from the design by Ahonen Q_1 and Q_2 are not made equal to keep the option for a Butterworth filter.

$$H_{hp4}(s) = \frac{KKs^4}{s^4 + s^3\left(\frac{\omega_0}{Q_1} + \frac{\omega_0}{Q_2}\right) + s^2\left(\omega_0^2 + \frac{\omega_0\omega_0}{Q_1 Q_2} + \omega_0^2\right) + s\left(\frac{\omega_0^2\omega_0}{Q_1} + \frac{\omega_0\omega_0^2}{Q_2}\right) + \omega_0^2\omega_0^2} \tag{4}$$

and further reduced to:

$$H_{hp4}(s) = \frac{K^2 s^4}{s^4 + s^3\left(\frac{(Q_1+Q_2)\omega_0}{Q_1 Q_2}\right) + s^2\left(\frac{(2Q_1Q_2+1)\omega_0^2}{Q_1 Q_2}\right) + s\left(\frac{(Q_1+Q_2)\omega_0^3}{Q_1 Q_2}\right) + \omega_0^4} \tag{5}$$

To obtain an expression for a high pass output, the definition if the transfer function is used:

$$H_{hp4}(s) = \frac{U_{hp}(s)}{U_{in}(s)} \tag{6}$$

By cross multiplying the nominator and denominator with U_{in} and U_{hp} :

$$U_{hp}(s)\left(s^4 + s^3\left(\frac{(Q_1+Q_2)\,\omega_0}{Q_1Q_2}\right) + e^2\left(\frac{(2Q_1Q_2+1)\,\omega_0^2}{Q_1Q_2}\right) + s\left(\frac{(Q_1+Q_2)\,\omega_0^3}{Q_1Q_2}\right) + \omega_0^4\right) = U_{in}(s)K^2s^4 \tag{7}$$

Then divide both sides by s^4:

$$U_{hp}(s)\left(1 + \left(\frac{(Q_1+Q_2)}{Q_1Q_2}\frac{\omega_0}{s}\right) + \left(\frac{(2Q_1Q_2+1)}{Q_1Q_2}\frac{\omega_0^2}{s^2}\right) + \left(\frac{(Q_1+Q_2)}{Q_1Q_2}\frac{\omega_0^3}{s^3}\right) + \frac{\omega_0^4}{s^4}\right) = U_{in}(s)K^2 \tag{8}$$

The State-Variable filter is synthesized using a summing circuit, and number of integrators which voltage is used for feedback with various attenuation factors. After reworking equation 8 the desired form is obtained:

$$\begin{aligned} U_{hp}(s) = U_{in}(s)K^2 - \frac{(Q_1+Q_2)}{Q_1Q_2}\left(\frac{\omega_0}{s}U_{hp}(s)\right) - \frac{(2Q_1Q_2+1)}{Q_1Q_2}\left(\frac{\omega_0^2}{s^2}U_{hp}(s)\right) - \\ \frac{(Q_1+Q_2)}{Q_1Q_2}\left(\frac{\omega_0^3}{s^3}U_{hp}(s)\right) - \frac{\omega_0^4}{s^4}U_{hp}(s) \end{aligned} \tag{9}$$

This form can easily be converted into a block diagram as in figure 1.

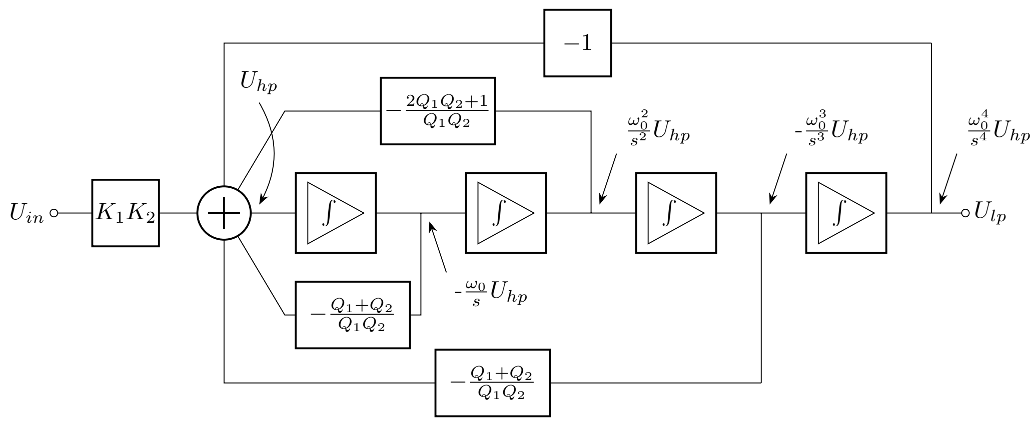


Figure 1: *Block diagram of a 4^{th} order State Variable filter*

Recalling equation 5 for the transfer function of a high-pass filter we can now write out the transfer function of the output of the fourth integrator in equation

10:

$$
\begin{aligned}
U_{lp} &= U_{hp} \cdot \frac{\omega_0}{s^4} \\
&= \frac{K^2 s^4}{s^4 + s^3\left(\frac{(Q_1+Q_2)\omega_0}{Q_1Q_2}\right) + s^2\left(\frac{(2Q_1Q_2+1)\omega_0^2}{Q_1Q_2}\right) + s\left(\frac{(Q_1+Q_2)\omega_0^3}{Q_1Q_2}\right) + \omega_0^4} \cdot \frac{\omega_0^4}{s^4} \\
&= \frac{K^2 \omega_0}{s^4 + s^3\left(\frac{(Q_1+Q_2)\omega_0}{Q_1Q_2}\right) + s^2\left(\frac{(2Q_1Q_2+1)\omega_0^2}{Q_1Q_2}\right) + s\left(\frac{(Q_1+Q_2)\omega_0^3}{Q_1Q_2}\right) + \omega_0^4}
\end{aligned}
\tag{10}
$$

This indeed is the expression for a 4^{th} order low pass filter.

4 Implementation

When the block diagram as in figure 1 is straightforwardly implemented a circuit as proposed by Ahonen results, figure 2.

The problem here is that the integrator outputs for the first and third order integrations $\frac{1}{s}$ and $\frac{1}{s^3}$ already are inverted with respect to the input signal and U_{hp}. Substracting these signals from the input signal U_{in} means that they have to be added in the summing stage by connecting them to the non-inverting input of the summing opamp OP1.

The non-inverting input of the opamp has a very high input impedance, hence the voltage at this summing point is a superposition of the input voltage and the two feedback signals. Every change in one of the resistors connected to this summing point will affect the amplification of all three signals. But this, in turn, means that every change in the feedback constants affects the other feedback constant as well as the overall gain of the input stage,

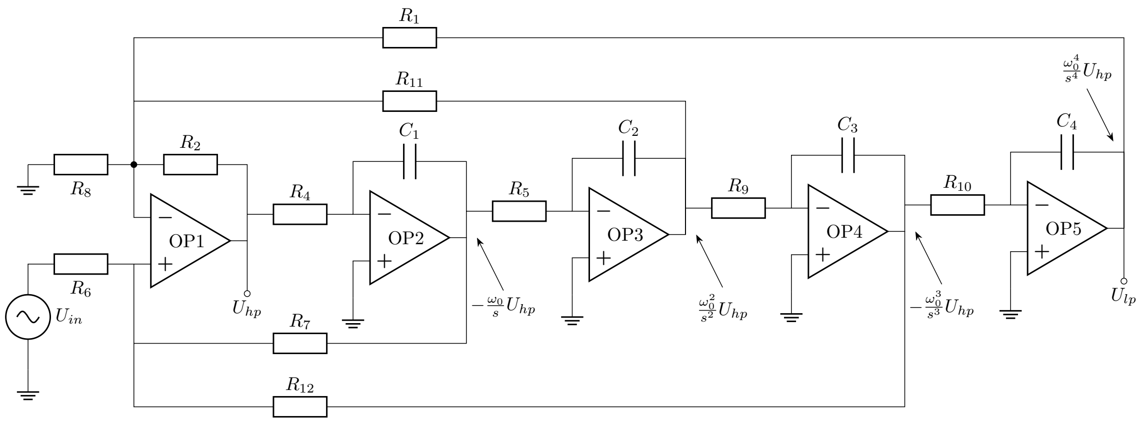


Figure 2: *Implementation using the non-inverting input of the summing amplifier as summing point*

The complexity of calculating the resistors in the non-inverting path can be overcome by adding one additional inverting amplifier OP6, figure 3. This

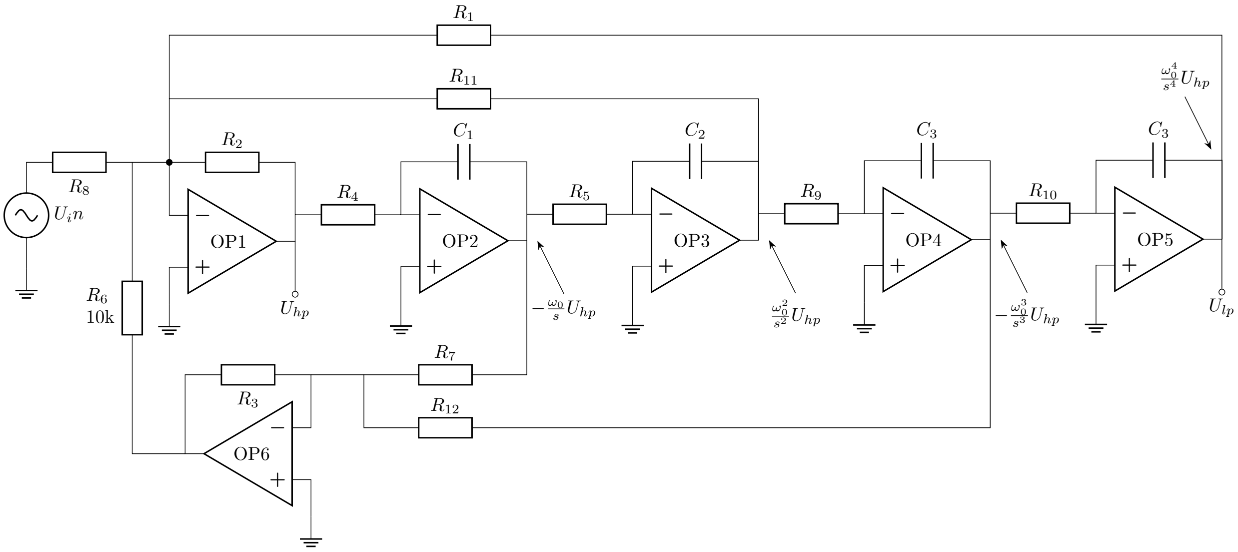


Figure 3: *Implementation with an additional inverter*

amplifier inverts the phase of the $\frac{1}{s}$ and $\frac{1}{s^3}$ terms. In this way these signals can be added to the inverting input of the summing amplifier.

Both terms have to be feed back to the summing amplifier with a gain of $\frac{Q_1+Q_2}{Q_1Q_2}$ so the gain of the inverter must be set to this value. The output signal of the inverter is summed with unity gain in the summing amplifier.

The signal input has also been moved to the inverting input of the summing amplifier. It is mandatory that the source has a low output impedance.

5 Circuit synthesis

5.1 Calculations

The component designation is according to figure 3.

When determining the component values it is assumed that R_2 is chosen first. This determines the gain of the summing amplifier. The value is not critical and values for the other resistors used for summation are derived using this value.

R_8: gain K^2:

$$K^2 = \frac{R_2}{R_8} \Rightarrow R_8 = \frac{R_2}{K^2} \tag{11}$$

R_6, R_3, R_7, R_{12} Gain of the inverting amplifier OP6 and feedback constant for $\frac{1}{s}$ and $\frac{1}{s^3}$: R_6 is set equal to R_2 so gain of this amplifier must be:

$$\frac{Q_1 + Q_2}{Q_1Q_2} \tag{12}$$

The feedback constant for both $\frac{1}{s}$ and $\frac{1}{s^3}$ is equal. By choosing an arbitrary value value for R_7, R_{12} the value for R_3 can be calculated.

$$\frac{R_3}{R_7, R_{12}} = \frac{Q_1 + Q_2}{Q_1 Q_2} \Rightarrow R_3 = R_7, R_{12} \frac{Q_1 + Q_2}{Q_1 Q_2} \tag{13}$$

R_1: Feedback constant for $\frac{1}{s^4}$ The feedback constant is -1 so the value for R_1 is equal to R_2.

R_{11}: Feedback constant for $\frac{1}{s^2}$ The feedback constant for this branch is:

$$\frac{R_2}{R_{11}} = \frac{2Q_1Q_2 + 1}{Q_1 Q_2} \Rightarrow R_{11} = R_2 \frac{Q_1 Q_2}{2Q_1 Q_2 + 1} \tag{14}$$

$R_4, C_1; R_5, C_2; R_9, C_3; R_{10}, C_4$ corner frequency ω_0 Each RC pair sets the integrator time constant

$$RC = \frac{1}{\omega_0} = \frac{1}{2\pi f_0} \tag{15}$$

The time constant must be equal for all integrator stages.

5.2 Example

In this example corner frequency is set at $f_0 = 185Hz$ and transfer characteristic is Linkwitz-Riley.

$\omega_0 = 2\pi \cdot f_0 = 2\pi \cdot 185 = 1162rad^{-1}$

Usually for capacitors less standard values are available than for resistors. In this case C1-C4 are set at 220nF.

$$RC = \frac{1}{\omega_0} \Rightarrow R = \frac{1}{1167 \cdot 220 \cdot 10^{-9}} = 3895\Omega$$

R_4, R_5, R_9 and R_{10} are set at $3.9k\Omega$

The value R_2 is quite arbitrary and normal design rules for opamp summing amplifiers should be followed. Here $R_2 = 10k\Omega$ is chosen.

For the application this circuit was built for a gain of approximately $10dB$ was needed. A gain of $10db$ is a gain factor of $10^{\frac{10}{20}}$ or approximately 3.2 times. From equation 11 we recall:

$$R_8 = \frac{R_2}{K^2} \Rightarrow R_8 = \frac{10 \cdot 10^3}{3.2} = 3.12k\Omega$$

A standard value of $3.3k\Omega$ will suffice

The value for R_6 is equal to R_2, in this case also $10k\Omega$

The gain factor for the feedback circuit through OP6 is set to unity by choosing $R_6 = R_1 = 10k\Omega$.

To determine R_3 first R_7 and R_{12} must be chosen. These values are non-critical and chosen is $R_7 = R_{12} = 10k\Omega$.

This example ia for a Linkwitz-Riley characteristic, hence $Q_1 = Q_2 = 0.707$. Then from equation 13:

$$R_3 = R_7, R_{12}\frac{Q_1 + Q_2}{Q_1 Q_2} \Rightarrow R_3 = 10 \cdot 10^3 \frac{0.707 + 0.707}{0.707 \cdot 0.707} = 28.4k\Omega$$

This value *is* critical and for actual circuit implementation the value must be chosen as close to this value as possible. For the simulation this exact value was used.

R_{11} from equation 14

$$R_{11} = R_2 \frac{Q_1 Q_2}{2Q_1 Q_2 + 1} = 10 \cdot 10^3 \frac{0.707 \cdot 0.707}{0.707 + 0.707} = 2.5k\Omega$$

6 Switch between Linkwitz-Riley or Butterworth

As mentioned previously in paragraph 3 if Q_1 and Q_2 are both 0.7 the 4^{th} order filter behaves as 2 cascaded Butterworth filters. However Q_1 and Q_2 can be set to the values which make up a real Butterworth 4^th order filter with an attenuation of 3dB at f_0.

From [3] we get the values $Q_1 = 0.543$ and $Q_2 = 1.31$ for 2 cascaded 2^{nd} order filters with a Butterworth transfer characteristic. To change the filter circuit, remarkable few changes have to be made. In table the Q-values, feedback constants and resistor values have been summarized.

	Q_1	Q_2	$\frac{Q_1+Q_2}{Q_1Q_2}$	$\frac{2Q_1Q_2+1}{Q_1Q_2}$	R_8	R_{10}
Butterworth	0.543	1.31	2.60	3.40	2.93k	26 k
Linkwitz-Riley	0.707	0.707	2.84	4.00	2.50k	28.4 k

Table 1: *Q-values and feedback constant for Butterworth and Linkwitz-Riley variants*

Unfortunately only Linkwitz-Riley and Butterworth filter characteristic have two 2^{nd} order sections which have the same cutoff frequency f_0. For other characteristics both Q_1 and Q_2 are different *and also* ω_{01} and ω_{02} from equation 3 are different. For implementation that would incur multiple integrator chains with multiple corner frequencies f_0. While certainly doable, it would void the elegance and simplicity of this circuit.

7 Simulation

The circuit of figure 3 has been entered in QUCS-S [2] and simulated. The QUCS diagram is according to the drawing on page 11. For the initial runs the theoretical resistance values from table 1 were used to prove the concept. A final run was made using the practical resistor values which were used to built the circuit with a Linkwitz-Riley characteristic. Although the State-Variable filter is not extremely sensitive to component tolerances, nevertheless seemingly small deviations from the theoretical values have a noticable impact. The Butterworth and Linkwitz-Riley response using the theoretical values are shown in 4 and 5. The signal levels and y-scales of the graphs have been modified to be the same as for the actual measurements. The reference level is set at $-47dB$ to accomodate for the reference level in the real measurement.

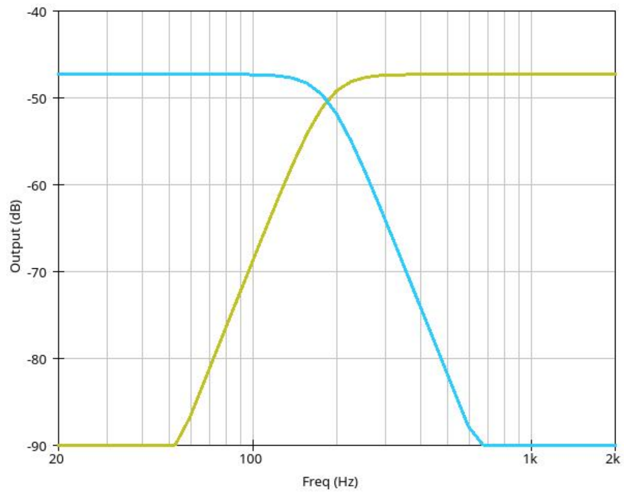


Figure 4: *Simulated transfer characteristic Butterworth adjustment* $f_0 = 185Hz$

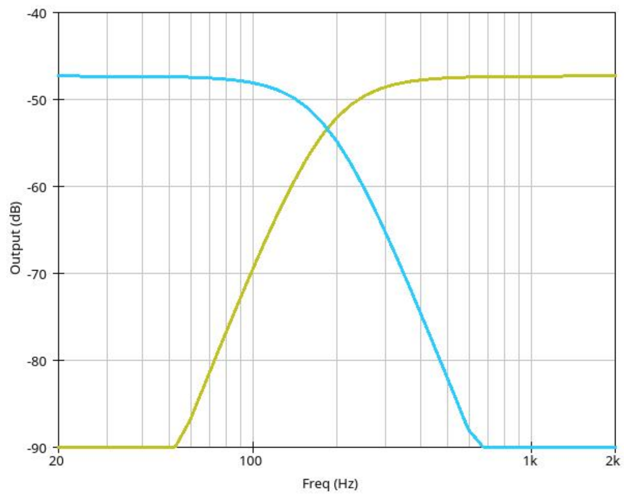
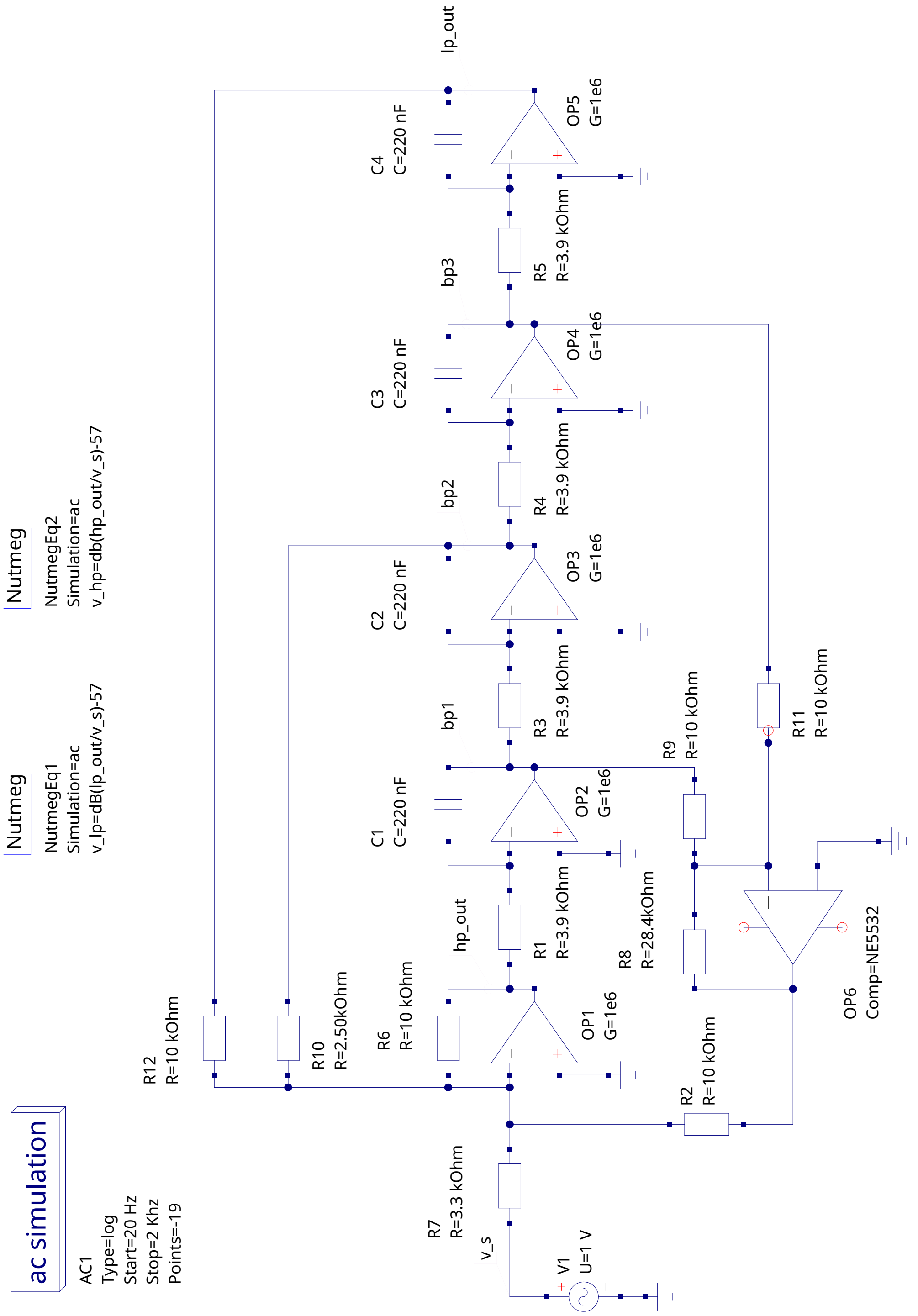


Figure 5: *Simulated transfer characteristic Linkwitz-Riley adjustment* $f_0 = 185Hz$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AC1
Type=log
Start=20 Hz
Stop=2 Khz
Points=19
Nutmeg
NutmegEq1
Simulation=ac
v_lp=dB(lp_out/v_s)-57
Nutmeg
NutmegEq2
Simulation=ac
v_hp=dB(hp_out/v_s)-57
V1
U=1 V
v_s
R7
R=3.3 kOhm
R12
R=10 kOhm
R10
R=2.50kOhm
R6
R=10 kOhm
OP1
G=1e6
R2
R=10 kOhm
hp_out
R1
R=3.9 kOhm
C1
C=220 nF
OP2
G=1e6
R8
R=28.4kOhm
R9
R=10 kOhm
OP6
Comp=NE5532
bp1
R3
R=3.9 kOhm
C2
C=220 nF
OP3
G=1e6
R11
R=10 kOhm
bp2
R4
R=3.9 kOhm
C3
C=220 nF
OP4
G=1e6
bp3
R5
R=3.9 kOhm
C4
C=220 nF
OP5
G=1e6
lp_out

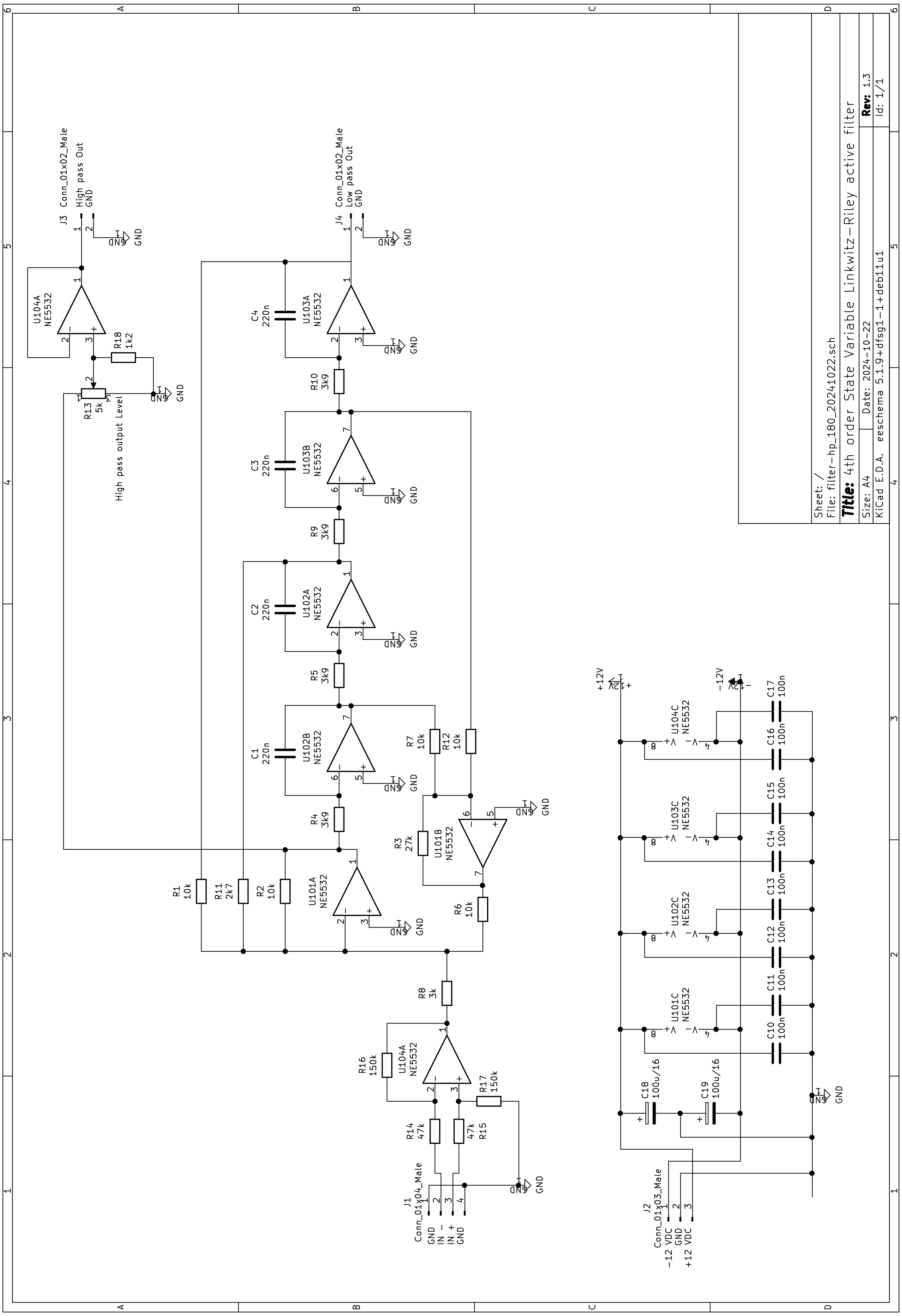
8 The actual circuit

The actual design of the filter part is unchanged with respect to the simulated version. Some trivial input and output buffers are added but these have no effect on the filter part.

The complete schematic is included on page 13. A differential input amplifier acts as the low impedance source to the input of the summing amplifier. The high-pass output is adjustable and buffered to provide a low-impedance output after the potentiometer. The low-pass output is intended as an input to another amplifier to sum Left and Right signals for a subwoofer. Therefor no additional gain adjusted output is added.

A nice touch is the addition of R_{18} to the contact of the adjustable resistor. It is very hard to obtain such an adjustable resistor in a logarithmic version. By adding R_{18} the output of the adjustable resistor is delinearized and somewhat resembles a logarithmic curve. Which is nice to have when controlling audio levels.

The circuit was built on breadboard as shipping costs for PCB services are prohibitively high for the area where I live. The breadboard layout is shown in figure 6. A picture of the assembled breadboard is in figure 7.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Size: A4
Date: 2024-10-22
Rev: 1.3
KiCad E.D.A. eeschema 5.1.9+dfsg1-1+deb11u1
Id: 1/1
J3 Conn_01x02_Male
High pass Out
GND
J4 Conn_01x02_Male
Low pass Out
GND
U104A NE5532
R18 1k2
R13 5k
High pass output Level
C4 220n
U103A NE5532
R10 3k9
C3 220n
U103B NE5532
R9 3k9
C2 220n
U102A NE5532
R5 3k9
C1 220n
U102B NE5532
R4 3k9
R7 10k
R12 10k
R3 27k
U101B NE5532
R6 10k
R1 10k
R11 2k7
R2 10k
U101A NE5532
R8 3k
R16 150k
U104A NE5532
R17 150k
R14 47k
R15 47k
J1 Conn_01x04_Male
GND
IN -
IN +
GND
+12V
-12V
U104C NE5532
U103C NE5532
U102C NE5532
U101C NE5532
C17 100n
C16 100n
C15 100n
C14 100n
C13 100n
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-12 VDC
GND
+12 VDC

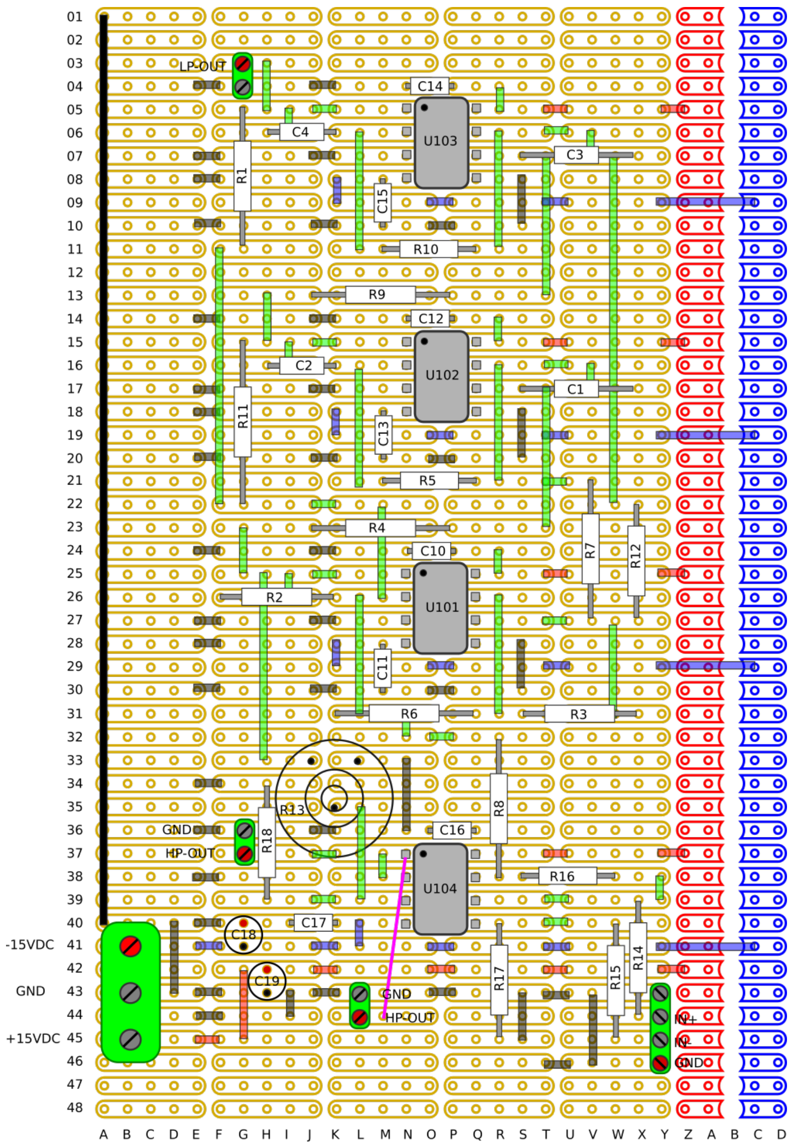

Figure 6: *Component layout as assembled on breadboard PCB*

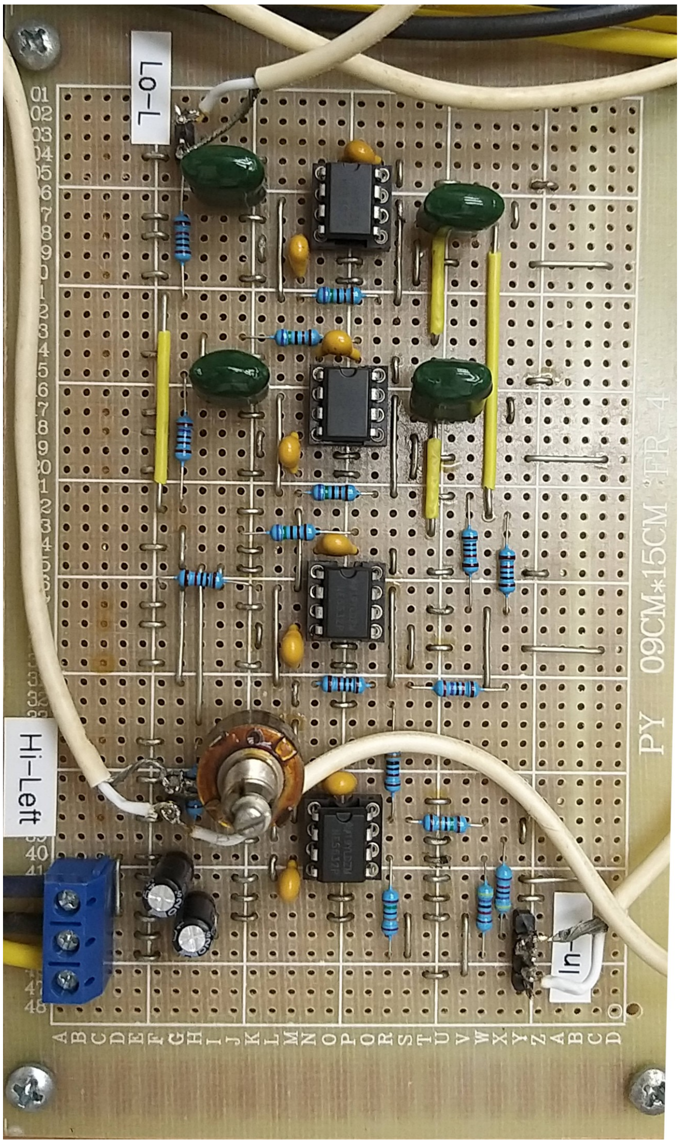

Figure 7: *Assembled breadboard PCB*

9 Measurement

A measurement was taken on simultaneously the High-pass and Low-pass outputs. The measured curve is shown in figure 8. The curves are nearly identical with the simulation result in figure 5. In the real world situation the damping at the crossover frequency is slightly less than -6dB. This is caused by choosing the resistors not exactly as calculated because of the availabilty of such values. In a later simulation it was obvious that this indeed causes a different damping at f_0.

The reference level of $-47dB$ is a result of the measurement circuit used. This value bears no relevance.

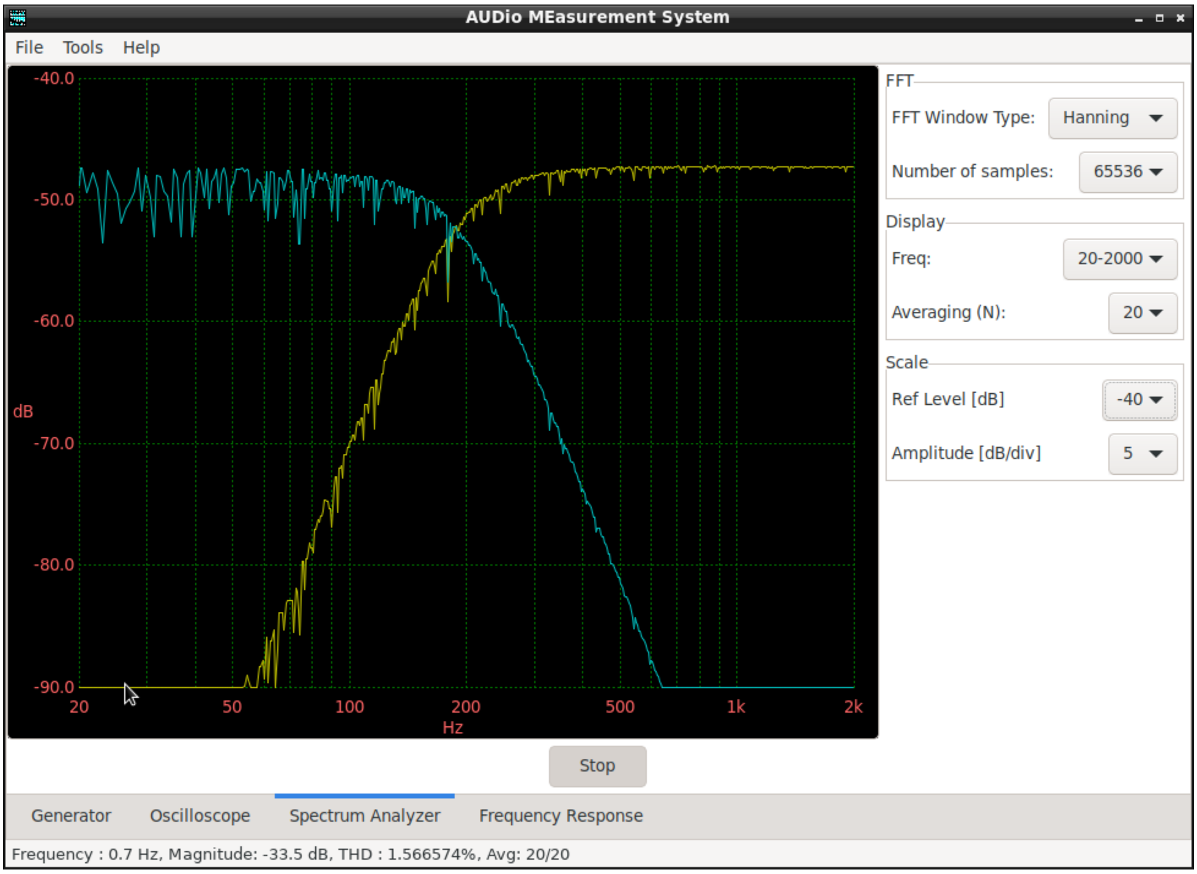

Figure 8: *Measurement on actual implementation of* 4^{th} *order Linkwitz-Riley filter.* $F_c = 185Hz$

References

[1] Janne Ahonen, *The analysis of fourth-order state variable filter and it's application to Linkwitz- Riley filters*, Linearteam tech paper, 2002

[2] Mike Brinson (mbrin72043@yahoo.co.uk) and Vadim Kusnetsov (ra3xdh@gmail.com), *Qucs-S Help documentation : User Manual and Reference Material*, https://qucs-s-help.readthedocs.io/en/latest/, 2018

[3] *Active Filter Cookbook*, Synergetic Press, New Mexico, 2nd edition, 1995