

On the Three-dimensional Corrections for One-dimensional Theory of Acoustic Horn

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Synopsis

In this paper three-dimensional corrections of ordinary one-dimensional theory of acoustic horns are discussed.

The results by means of the similar perturbation method adopted by the author to solve the problem of longitudinal stress waves in a bar of variable cross-section are given and compared with A. F. Stevenson's exact solutions.

And a numerical example in the case of a exponential horn including the 6th order approximation is added.

1. Introduction

To estimate the various characteristics of acoustic horns of different shapes, it is usual to start from the approximate one-dimensional equation of motion¹⁾, in which sound waves in the treated horn are assumed to be plane and sound intensity is uniform distributed over the cross-section perpendicular to the axis of the horn. This approximation is said to be sufficiently accurate for horns of usual shape, when the flaring angle of the horn is sufficiently small and the wave length of propagating sound waves is not so small compared with the linear dimensions of its cross-section.

But, in view of its widely spread use and the great importance of its practical applications, it may be worthwhile to estimate the errors in this approximate procedure, or, advancing a step further, to evaluate as accurately as possible the velocity potential of the propagating waves at any point in the horn, from which its various characteristics, especially the directional characteristics of the outside radiating waves can be calculated.

Concerning this point, the rigorous theory of acoustic horns of any shape established by A. F. Stevenson²⁾ shed much light on these problems. This theory based on the theory of the same author on the electro-magnetic horns³⁾. According to this theory, the velocity potential ϕ is expanded into infinite series, each term of which is an orthogonal eigen-function satisfying the Poisson's equation $\nabla^2\phi + k^2\phi = 0$ in the domain of the cross-section and the boundary condition that makes its derivative along the normal to the boundary curve of the cross-section zero, $\frac{\partial\phi}{\partial\nu} = 0$, multiplied by the suitable function of z as the coefficient of expansion, z being the distance measured along the axis of the horn. These functions of z , infinite in number, can be determined, this point being the crux of the theory, by an infinite set of simultaneous ordinary differential equations. Though this theory is formally complete as a solution of the problem, it is very cumbersome, if not impossible, to accomplish the actual computation of the velocity potential according to this theory. Stevenson resorted to W. K. B. method and estimate the effect of flare up to the order of θ^2 , θ being the flaring angle.

Confronting the similar problems of solid horns which are widely used in ultrasonic carving machine, the author of this paper had tried to apply the usual perturbation method to estimate the errors committed by the ordinary theory of the axisymmetric longitudinal stress waves in the solid bar of variable circular cross-section⁴⁾. This procedure consists in expanding the functions expressing the three mutually perpendicular displacements of any point in the body

into the double infinite series of ascending powers of two parameters assumed small, which are the flaring angle and the ratio of the length of the cross-sectional dimension to the wave length, and in determining the coefficient of each term by substituting this series in the three-dimensional general equations of motion of elastic body and the equations expressing boundary conditions, and by equating the sum of terms of the same powers on two sides of the equations.

This method, as the process above mentioned clearly shows, can be easily transferred without important alternations to the problems of acoustic horns⁵⁾. Instead of its poor generality treating only a special case of circular cross-section and the principal mode⁶⁾ of axisymmetric propagating waves, the accuracy of its approximation can be easily raised up to higher orders of the two small quantities above mentioned. (in this paper results up to 6th order approximation are given.)

But tracing back to the starting point, Stevenon's method being the most general in character, the results obtained by the author's method ought to have been deduced from his method too, and it may not be without any value to give in the following this deduction briefly.

2. Results from Direct Perturbation Method

We choose the cylindrical polar coordinate system (z, r, ϕ) , whose z -axis coincides with that of the horn. (Fig. 1) The cross-section A of the horn perpendicular to its axis are all circles C_r

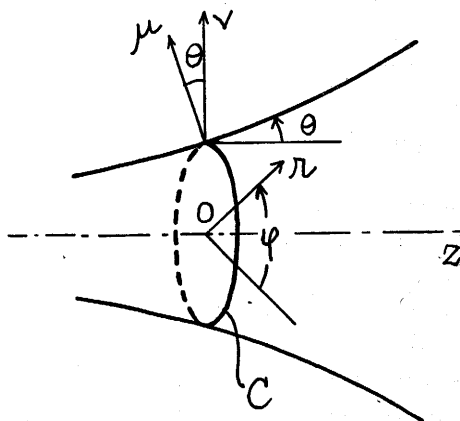


Fig. 1 General View of Horn

whose centres lie on the axis and whose radii a are expressed by an specified function $a(z)$, as an example the exponential function $a = a_0 \exp(\mu z)$ in the case of exponential horn.

We consider only the steady state propagation and waves of a particular circular frequency ω and accordingly omit the common time factor $\exp(i\omega t)$ from the velocity potential ϕ and other quantities. The velocity potential ϕ must satisfy the Poisson's equation

$$\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0, \quad \dots\dots\dots(2,01)$$

where ∇^2 stands for

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

and

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad \dots\dots\dots(2,02)$$

c being the sound velocity in air and λ being the wave length corresponding to the frequency $(\omega/2\pi)$. The velocity potential ϕ must also satisfy the boundary condition

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial z} \tan \theta = \frac{da}{dz} \cdot \frac{\partial \phi}{\partial z} \quad \dots\dots\dots(2,03)$$

on the circumference C of the cross-section, θ being the flaring angle at that position z .

Furthermore, we confine ourselves to only principal mode⁽¹⁾ of propagation, which is axisymmetric and has no nodal circle over the cross-section, and express the velocity potential in this case in the form of infinite series, :

$$\phi = \phi^{(0)} + \phi^{(2)} + \phi^{(4)} + \dots, \quad (2,04)$$

where each term $\phi^{(2n)}$ ($n=1, 2, 3, \dots$) is in turn the sum of terms having order of magnitude equal to $2n$ -th order of the two small parameters $\tan\theta$ and ka . (We assume tentatively these two parameters to be of the same order of smallness. If they differ from each other, we can rearrange the terms in the series and collect together the terms of the same order of smallness into one group.)

By introducing (2,04) into (2,01) and (2,03) and equating the terms of the same order of smallness, namely those of the same powers of $\tan\theta$ and ka , in both sides of the equations, we obtain following infinite set of equations determining $\phi^{(0)}$, $\phi^{(2)}$, $\phi^{(4)}$,..... In this operation, we take into consideration that differentiation with respect to r or dividing by r diminishes the order by one and multiplying by $\tan\theta$ or (da/dz) increase it by one.

Namely, from (2,01)

$$\frac{\partial^2 \phi^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \phi^{(0)}}{\partial r} = 0 \quad (2,05)(i)$$

$$\frac{\partial^2 \phi^{(2)}}{\partial r^2} + \frac{1}{r} \frac{\partial \phi^{(2)}}{\partial r} = - \left\{ \frac{\partial^2 \phi^{(2)}}{\partial z^2} + k^2 \phi^{(0)} \right\} \quad (2,05)(ii)$$

$$\frac{\partial^2 \phi^{(4)}}{\partial r^2} + \frac{1}{r} \frac{\partial \phi^{(4)}}{\partial r} = - \left\{ \frac{\partial^2 \phi^{(2)}}{\partial z^2} + k^2 \phi^{(2)} \right\} \quad (2,05)(iii)$$

$$\vdots \quad \vdots$$

and from (2,03), on the boundary C ,

$$\frac{\partial \phi^{(0)}}{\partial r} = 0 \quad (2,06)(i)$$

$$\frac{\partial \phi^{(2)}}{\partial r} = \frac{da}{dz} \frac{\partial \phi^{(0)}}{\partial z} \quad (2,06)(ii)$$

$$\frac{\partial \phi^{(4)}}{\partial r} = \frac{da}{dz} \frac{\partial \phi^{(2)}}{\partial z} \quad (2,06)(iii)$$

$$\vdots \quad \vdots$$

These equations can be solved step by step, and at first from (2,05)(i) we obtain

$$\phi^{(0)} = \psi_0(z) \log r + \Phi_0(z)$$

where $\Phi_0(z)$ and $\psi_0(z)$ are two arbitrary function of integration and $\psi_0(z)$ must be put to zero when on the z -axis there are neither sources nor sinks, and $\phi^{(0)}$ becomes

$$\phi^{(0)} = \Phi_0(z) \quad (2,07)$$

Introducing this in (2,05)(ii) and (2,06)(ii),

$$\frac{\partial^2 \phi^{(2)}}{\partial r^2} + \frac{1}{r} \frac{\partial \phi^{(2)}}{\partial r} = -(\Phi_0'' + k^2 \Phi_0) \quad (2,08)$$

$$\text{and} \quad \frac{\partial \phi^{(2)}}{\partial r} = \frac{da}{dz} \Phi_0' \quad \text{at} \quad r=a \quad (2,09)$$

where the accentuation, denotes differentiation with respect to z . From (2,08), we obtain by omitting again the term becoming infinite on the z -axis

$$\frac{\partial \phi^{(2)}}{\partial r} = -(\Phi_0'' + k^2 \Phi_0) \frac{r}{2} \quad (2,10)$$

and (2,09) becomes

$$\Phi_0'' + \frac{2}{a} \frac{da}{dz} \Phi_0' + k^2 \Phi_0 = 0 \quad (2,11)$$

This equation (2.11) is the same to the equation satisfied by the velocity potential of the approximate one-dimensional theory, and accordingly $\phi^{(0)}$ may be termed as the zero-order approximation.

Integrating (2.10) with respect to r , and considering (2.11), $\phi^{(2)}$ can be written in the form,

$$\phi^{(2)} = \Phi_2(z) + \frac{1}{a} \frac{da}{dz} \Phi_0' \frac{r^2}{2}, \quad \dots\dots\dots(2.12)$$

$\Phi_2(z)$ being an arbitrary function of integration yet to be determined. This form of $\phi^{(2)}$ substituted into (2.05)(iii) gives

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi^{(2)}}{\partial r} \right) = -(\Phi_2'' + k^2 \Phi_2) + (\Phi_0' \frac{d}{dz} + 2\Phi_0'') \left\{ \left(\frac{1}{a} \frac{da}{dz} \right)^2 - \frac{\partial}{\partial z} \left(\frac{1}{a} \frac{da}{dz} \right) \right\} \frac{r^2}{2}.$$

and integrating this and omitting again the term becoming infinitely large on the z -axis, we obtain

$$\frac{\partial \phi^{(2)}}{\partial r} = -(\Phi_2'' + k^2 \Phi_2) \frac{r}{2} + (\Phi_0' \frac{d}{dz} + 2\Phi_0'') \left\{ \left(\frac{1}{a} \frac{da}{dz} \right)^2 - \frac{\partial}{\partial z} \left(\frac{1}{a} \frac{da}{dz} \right) \right\} \frac{r^3}{8} \quad \dots\dots\dots(2.13)$$

With (2.06)(iii) this gives

$$\Phi_2'' + \frac{2}{a} \frac{da}{dz} \Phi_2' + k^2 \Phi_2 = -\frac{1}{4} \left(\Phi_0' \frac{d}{dz} + 2\Phi_0'' \right) \left\{ \left(\frac{1}{a} \frac{da}{dz} \right)^2 - \frac{d}{dz} \left(\frac{1}{a} \frac{da}{dz} \right) \right\} \quad \dots\dots\dots(2.14)$$

These equation determines Φ_2 , which in turn introduced into (2.12) constitute the 2nd order correction $\phi^{(2)}$ for the potential. Proceeding quite similarly, we can determine higher order corrections one after another successively. The infinite series (2.04) is semi-convergent one and constitute an asymptotic expansion for the velocity potential.

As an example, we will treat the case of a exponential horn, whose radius a of the cross-section z is expressed by

$$a = a_0 \exp(\mu z), \quad \dots\dots\dots(2.15)$$

a_0 being the radius at $z=0$ and μ being the flaring index.

Writing down only the results obtained by the above mentioned procedures, the corrections of velocity potential up to 6th order approximation become as follows : —

$$\begin{aligned} \phi^{(0)} &= \exp\{(-\mu + iq)z\} \\ \phi^{(2)} &= -\left\{ \frac{-\mu + iq}{8(\mu + iq)} a^2 - \frac{r^2}{2} \right\} \mu(-\mu + iq) \exp\{(-\mu + iq)z\} \\ \phi^{(4)} &= \left\{ -\frac{5\mu + 13iq}{384(2\mu + iq)} a^4 + \frac{1}{16} a^2 r^2 + \frac{1}{16} r^4 \right\} \mu^2(-\mu + iq)^2 \exp\{(-\mu + iq)z\} \\ \phi^{(6)} &= \left[-\frac{251\mu^3 - 765\mu q^2 + i(913\mu^2 q - 183q^3)}{9216(2\mu + iq)(3\mu + iq)} a^6 \right. \\ &\quad \left. + \frac{(5\mu + 13iq)(5\mu + 3iq)}{768(2\mu + iq)} a^4 r^2 - \frac{\mu + iq}{128} a^2 r^4 + \frac{-\mu + iq}{288} r^6 \right] \mu^3(-\mu + iq)^2 \exp\{(-\mu + iq)z\} \\ &\vdots \quad \quad \quad \vdots \\ &\quad (i = \sqrt{-1}) \quad \quad \quad \dots\dots\dots(2.16) \end{aligned}$$

where q^2 stands for

$$q^2 = k^2 - \mu^2 \quad \dots\dots\dots(2.17)$$

and a constant factor and the time factor $\exp(i\omega t)$ are both omitted from each $\phi^{(2n)}$.

The velocity potential built up from these according to (2.04) gives the total energy flux the same at all cross-sections along the z -axis and corresponds to waves propagating in the positive z -direction, while those propagating in the negative z -direction can be represented by substituting $-q$ in place of $+q$ in (2.16).

Fig. 2 shows the sound energy flux distribution over the various cross-sections of a infinite

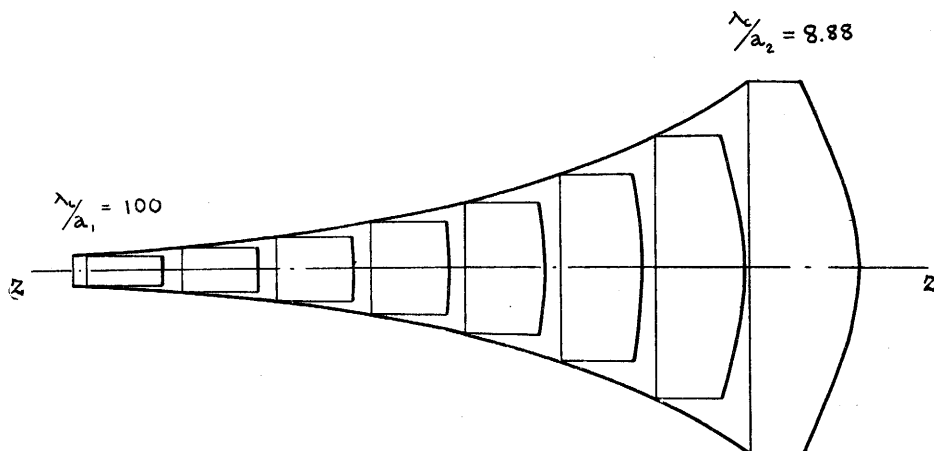


Fig. 2 Sound energy flux distribution E_v / \bar{E}_v over various cross-sections of a exponential horn ($f/f_c=3.0$)
 (f : frequency of propagating waves, f_c : cut-off frequency
 (λ_c : cut-of wave length, a_1, a_2 : radii of cross-section at throat and mouth)

long exponential horn, in which the waves of the frequency f three times as great as its cut-off frequency f_c propagate in the positive z -direction. In this figure the energy flux parallel to z -axis per unit area of the cross-section E_v divided by their mean value \bar{E}_v is represented.

3. Short Recapitulation of Stevenson's Method and Deduction by it of the Results in Last Paragraph

Let v_n be the normalized and orthogonal eigen-functions with eigenvalues β_n ($n=0, 1, 2, 3, \dots$) defined by

$$(\nabla^2 + \beta_n^2)v_n = 0, \quad \dots\dots\dots(3.01)$$

in the cross-section A and

$$\frac{\partial v_n}{\partial \nu} = 0 \quad \dots\dots\dots(3.02)$$

on the boundary C, of which ν is the normal.

We next assume the following eigen-function expansion of the velocity potential ϕ ,

$$\phi = \sum_{n=0}^{\infty} \phi_n(z) v_n$$

According to A.F. Stevenson, the functions $\phi_n(z)$ appearing as the coefficients of expansion can be determined by an infinite set of simultaneous differential equations

$$\phi_n'' + \sum_{m=0}^{\infty} X_{nm} \phi_m' + \sum_{m=0}^{\infty} Y_{nm} \phi_m = 0, \quad n=0, 1, 2, 3, \dots, \quad \dots\dots\dots(3.04)$$

The coefficients X_{nm} and Y_{nm} in this equation are defined by

$$X_{nm} = A_{nm} - A_{mn} \quad \dots\dots\dots(3.05)$$

$$Y_{nm} = (k^2 - \beta_n^2) \delta_{mn} + C_{nm} + D_{nm} \quad \dots\dots\dots(3.06)$$

where the symbol δ_{mn} is Klonecker's delta, namely $=1$ when $n=m$, and $=0$ when $n \neq m$, and A_{mn} , C_{mn} and D_{nm} stand for the following integrals : —

$$A_{mn} = \int_A v_m \frac{\partial v_n}{\partial z} dS, \quad C_{mn} = \int_A v_m \frac{\partial^2 v_n}{\partial z^2} dS, \quad \dots\dots\dots(3.07)$$

$$D_{mn} = \int_C v_m \frac{\partial v_n}{\partial z} \tan \theta \, ds, \quad \dots\dots\dots (3.08)$$

ds being the elementary area of A , and ds being the line element of C .

The normalized eigen-function v_n in our case of circular cross-section is

$$v_n = \frac{1}{a\sqrt{\pi}} \frac{J_0(x_n \frac{r}{a})}{J_0(x_n)}, \quad \dots\dots\dots (3.09)$$

where x_n is the n -th of infinite number of roots of the equation

$$J_1(x_n) = 0, \quad \dots\dots\dots (3.10)$$

the first of them being $x_0 = 0$.

Corresponding to these, the eigenvalues β_n are given by

$$\beta_n = \frac{x_n}{a} \quad \dots\dots\dots (3.11)$$

and thus inversely proportional to the variable radius of the cross-sectional circle.

Differentiating (3.09) with respect to z , we obtain

$$\frac{\partial v_n}{\partial z} = - \frac{(da/dz)}{a^2\sqrt{\pi}} \frac{1}{J_0(x_n)} \{J_0(\beta_n r) + (\beta_n r)J_0'(\beta_n r)\} \quad \dots\dots\dots (3.12)$$

This derivatives of v_n can again be expressed by the eigenfunction expansion as

$$\frac{\partial v_n}{\partial z} = -\mu \sum_{m=1}^{\infty} \alpha_{nm} v_m, \quad \dots\dots\dots (3.13)$$

μ being given as

$$\mu = \frac{1}{a} \frac{da}{dz} \quad \dots\dots\dots (3.14)$$

and constant in the case of exponential horn treated in the following.

The coefficients α_{nm} of this expansion are determined according to deductions in Appendix A as,

$$\left. \begin{aligned} \alpha_{n0} &= 2, \quad \alpha_{0n} = 0, \quad \alpha_{00} = 1, \quad n \neq 0, \quad m \neq 0 \\ \alpha_{nm} &= 1, \quad \alpha_{nm} = \frac{2x_n^2}{x_n^2 - x_m^2}, \quad m \neq n \end{aligned} \right\} \quad \dots\dots\dots (3.15)$$

when m and n are not equal to zero and differ from each other.

With these results, A_{nm} , C_{nm} and D_{nm} in (3.07) and (3.08) can be written down by (3.05) and (3.06) as

$$A_{nm} = -\mu \alpha_{nm} \quad \dots\dots\dots (3.16)$$

$$\left. \begin{aligned} C_{nm} &= C_{mn} = -4\mu^2 \left\{ 1 + \sum_{l=1}^{\infty} \frac{x_n^2 x_m^2}{(x_n^2 - x_l^2)(x_m^2 - x_l^2)} \right\} \\ &= -8\mu^2 \frac{x_n^2 x_m^2}{(x_n^2 - x_m^2)^2} \end{aligned} \right\} \quad \dots\dots\dots (3.17)$$

$$C_{nn} = -\mu^2 \left(3 + 4 \sum_{l=1}^{\infty} \frac{x_n^4}{(x_n^2 - x_l^2)^2} \right) = \mu^2 \left(1 - \frac{x_n^2}{3} \right)$$

$$C_{n0} = C_{0n} = 0, \quad C_{00} = \mu^2 \quad m \neq n, \quad m \neq 0, \quad n \neq 0$$

the deductions of these results being given in Appendix A and B.

Similarly

$$D_{nm} = D_{mn} = D_{nn} = D_{n0} = D_{0n} = D_{00} = -2\mu^2 \quad \dots\dots\dots (3.18)$$

The notation of summation above used \sum' means the sum of all terms with the exception of that corresponding to $l=n$, and \sum'' with the exception of those corresponding to $l=m$ and $l=n$.

Utilizing (3.16) (3.17) and (3.18), X_{nm} and Y_{nm} defined in (3.05) and (3.06) can be determined as

$$\left. \begin{aligned} X_{n0} &= 2\mu, \quad X_{0n} = -2\mu, \quad X_{00} = 0 \quad n \neq 0, \quad m \neq 0 \\ X_{nm} &= 2\mu \frac{x_n^2 + x_m^2}{x_n^2 - x_m^2}, \quad X_{nn} = 0 \quad n \neq m \end{aligned} \right\} \quad \dots\dots\dots (3.19)$$

$$\text{and } \left. \begin{aligned} Y_{n0} &= Y_{0n} = -2\mu^2, \quad Y_{00} = k^2 - \mu^2 = q^2, \\ Y_{nm} &= -2\mu^2 \left\{ 1 + \frac{4x_n^2 x_m^2}{(x_n^2 - x_m^2)^2} \right\} \quad n \neq 0, \quad m \neq 0, \quad n \neq m \\ Y_{nn} &= k^2 - \mu^2 - \beta_n^2 - \frac{\mu^2}{3} x_n^2 \end{aligned} \right\} \dots\dots\dots (3,20)$$

And these coefficients being known, the infinite set of simultaneous differential equations determining ϕ_n ($n=0, 1, 2, \dots$) can be written down as follows : —

$$\phi_0'' + (k^2 - \mu^2)\phi_0 - 2\mu \sum_{m=1}^{\infty} \phi_m' - 2\mu^2 \sum_{m=1}^{\infty} \phi_m = 0 \quad \dots\dots\dots (3,21)$$

$$\begin{aligned} \phi_n'' + (k^2 - \mu^2 + \beta_n^2 - \frac{\mu^2}{3} x_n^2) \phi_n + 2\mu \phi_0' - 2\mu^2 \phi_0 \\ + 2\mu \sum_{m=1}^{\infty} \frac{x_n^2 + x_m^2}{x_n^2 - x_m^2} \phi_m' - 2\mu^2 \sum_{m=1}^{\infty} \left[1 + \frac{4x_n^2 x_m^2}{(x_n^2 - x_m^2)^2} \right] \phi_m = 0, \quad n \neq 0 \end{aligned} \quad \dots\dots\dots (3,22)$$

To solve these equations, we resort to the usual perturbation method and expand ϕ_n into the sum of successive terms of the same power of two small parameters, $\mu a = \tan \theta$ and ka , namely

$$\phi_n = \phi_{n0} + \phi_{n2} + \phi_{n4} + \dots \quad (n=0, 1, 2, \dots) \quad \dots\dots\dots (3,23)$$

By introducing (3,23) into (3,21) and (3,22) and equating the terms of the same power of two parameters, we can obtain the following infinite set of differential equations : —

$$\left. \begin{aligned} \phi_{00}'' + (k^2 - \mu^2)\phi_{00} &= 0 \\ \phi_{02}'' + (k^2 - \mu^2)\phi_{02} &= 2\mu \sum_{m=1}^{\infty} \phi_{m2}' + 2\mu^2 \sum_{m=1}^{\infty} \phi_{m2} \\ \phi_{04}'' + (k^2 - \mu^2)\phi_{04} &= 2\mu \sum_{m=1}^{\infty} \phi_{m4}' + 2\mu^2 \sum_{m=1}^{\infty} \phi_{m4} \\ &\vdots \\ &(n \neq 0) \end{aligned} \right\} \quad \dots\dots\dots (3,24)$$

$$\text{and } \left. \begin{aligned} \beta_n^2 \phi_{n2} &= 2\mu \phi_{00}' - 2\mu^2 \phi_{00} \\ \beta_n^2 \phi_{n4} &= 2\mu \phi_{02}' - 2\mu^2 \phi_{02} + \phi_{n2}'' + (k^2 - \mu^2 - \frac{x_n^2}{3} \mu^2) \phi_{n2} \\ &\quad + 2\mu \sum_{m=1}^{\infty} \frac{x_n^2 + x_m^2}{x_n^2 - x_m^2} \phi_{m2}' - 2\mu^2 \sum_{m=1}^{\infty} \left(\frac{x_n^2 + x_m^2}{x_n^2 - x_m^2} \right)^2 \phi_{m2} \\ \beta_n^2 \phi_{n6} &= 2\mu \phi_{04}' - 2\mu^2 \phi_{04} + \phi_{n4}'' + (k^2 - \mu^2 - \frac{x_n^2}{3} \mu^2) \phi_{n4} \\ &\quad + 2\mu \sum_{m=1}^{\infty} \frac{x_n^2 + x_m^2}{x_n^2 - x_m^2} \phi_{m4}' - 2\mu^2 \sum_{m=1}^{\infty} \left(\frac{x_n^2 + x_m^2}{x_n^2 - x_m^2} \right)^2 \phi_{m4} \\ &\vdots \\ &(n \neq 0) \end{aligned} \right\} \quad \dots\dots\dots (3,25)$$

For waves propagating in the positive z -direction, we start with

$$\phi_{00} = \exp(iqz), \quad q^2 = k^2 - \mu^2 \quad \dots\dots\dots (3,26)$$

which gives

$$\phi_0 = \frac{1}{a\sqrt{\pi}} \exp(iqz) = \frac{1}{a_0\sqrt{\pi}} \exp\{(-\mu + iq)z\} \quad \dots\dots\dots (3,27)$$

as the zero-order approximation, which coincide with the results of the usual one-dimensional theory.

Then from the first equation of (3,25), we obtain

$$\phi_{n2} = \frac{2a^2}{x_n^2} \mu (-\mu + iq) \exp(iqz) \quad \dots\dots\dots (3,28)$$

and introducing this into the first equation of (3,24)

$$\phi_{02} = \frac{a^2}{8} \cdot \frac{\mu(-\mu + iq)(3\mu + iq)}{\mu + iq} \exp(iqz) \quad \dots\dots\dots (3,29)$$

Thus the correction term of 2nd order becomes

$$\phi^{(2)} = \sum_{n=0}^{\infty} \phi_{n2} v_n = \frac{\mu(-\mu+iq)}{a\sqrt{\pi}} \left[\frac{1}{8} \frac{3\mu+iq}{\mu+iq} + \sum_{n=1}^{\infty} \frac{2}{x_n^2} \cdot \frac{J_n(\beta_n r)}{J_n(x_n)} \right] a^2 \exp(iqz) \quad (3,30)$$

and by utilizing the expansion (B,07) in Appendix B, this can be arranged into

$$\phi^{(2)} = -\frac{1}{a_0\sqrt{\pi}} \left[\frac{-\mu+iq}{8(\mu+iq)} a^2 - \frac{r^2}{8} \right] \mu(-\mu+iq) \exp\{(-\mu+iq)z\} \quad (3,31)$$

Proceeding in the similar manner, the 4th order correction terms can be determined as

$$\phi_{n4} = \left(\frac{3}{4} \frac{1}{x_n^2} - \frac{4}{x_n^4} \right) \mu^2(-\mu+iq)^2 a^4 \exp(iqz) \quad (3,32)$$

$$\phi_{04} = \frac{7(5\mu+iq)}{384(2\mu+iq)} \mu^2(-\mu+iq)^2 a^4 \exp(iqz) \quad (3,33)$$

and these being combined

$$\phi^{(4)} = \sum_{n=0}^{\infty} \phi_{n4} v_n = \frac{\mu^2(-\mu+iq)^2}{a\sqrt{\pi}} \left[\frac{7(5\mu+iq)}{384(2\mu+iq)} + \sum_{n=1}^{\infty} \left(\frac{3}{4} \frac{1}{x_n^2} - \frac{4}{x_n^4} \right) \frac{J_n(\beta_n r)}{J_n(x_n)} \right] \times a^4 \exp(iqz) \quad (3,34)$$

Utilizing again the expansions (B,07) and (B,10), the 4th order correction term $\phi^{(4)}$ can be arranged into

$$\phi^{(4)} = \frac{1}{a_0\sqrt{\pi}} \left[-\frac{5\mu+13iq}{384(2\mu+iq)} a^4 + \frac{1}{16} a^2 r^2 + \frac{1}{16} r^4 \right] \mu^2(-\mu+iq)^2 \exp\{(-\mu+iq)z\} \quad (3,35)$$

Similarly,

$$\phi_{n6} = \left[\left\{ \frac{7(3\mu+iq)(13\mu+5iq)}{192(2\mu+iq)} - \frac{3}{4} \mu \right\} \frac{1}{x_n^2} + \left(\frac{3}{2} \mu - \frac{1}{2} iq \right) \frac{1}{x_n^4} + \frac{8(-\mu+iq)}{x_n^6} \right] a^6(-\mu+iq)^2 \mu^3 \exp(iqz) \quad (3,36)$$

$$\phi_{06} = \frac{7\mu+iq}{6(3\mu+iq)} \left[\frac{7}{8} \frac{(3\mu+iq)(13\mu+iq)}{192(2\mu+iq)} - \frac{17}{192} \mu \right] a^6(-\mu+iq)^2 \mu^3 \exp(iqz) \quad (3,37)$$

and after some calculations utilizing the expansions (B,07), (B,10) and (B,12) in Appendix B, we obtain

$$\begin{aligned} \phi^{(6)} &= \sum_{n=0}^{\infty} \phi_{n6} v_n = \frac{1}{a\sqrt{\pi}} \left(\phi_{06} + \sum_{n=1}^{\infty} \phi_{n6} \frac{J_0(\beta_n r)}{J_0(x_n)} \right) \\ &= \frac{1}{a_0\sqrt{\pi}} \left[-\frac{(251\mu^3-765\mu q^2+i(913\mu^2 q-183q^3))}{9216(2\mu+iq)(3\mu+iq)} a^6 \right. \\ &\quad \left. + \frac{(5\mu+13iq)(5\mu+3iq)}{768(2\mu+iq)} a^4 r^2 - \frac{\mu+iq}{128} a^2 r^4 + \frac{-\mu+iq}{283} r^6 \right] \times \\ &\quad \times \mu^3(-\mu+iq)^2 \exp\{(-\mu+iq)z\} \quad (3,38) \end{aligned}$$

The results thus obtained (3,31) (3,35) and (3,38) coincide with the former ones in (2,16) excepting constant factors.

4. Acknowledgement

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Appendix A

i) The coefficients α_{nm} in the expansion (3,13) are evaluated by the following integral when m and n are both integer not equal to zero and differ from each other, namely

$$\alpha_{nm} = \frac{1}{\pi a^2 J_0(x_m) J_0(x_n)} \int_0^a \beta_n r J_0'(\beta_n r) J_0(\beta_m r) 2\pi r dr \quad (A,01)$$

and accordingly they satisfy the following two relations, one of which is

$$\alpha_{nm} + \alpha_{mn} = \frac{1}{\pi a^2 J_0(x_n) J_0(x_m)} \int_0^a \frac{d}{dr} \{J_0(\beta_n r) J_0(\beta_m r)\} 2\pi r^2 dr = 2, \quad \text{.....(A,02)}$$

and the other is

$$\beta_m^2 \alpha_{nm} + \beta_n^2 \alpha_{mn} = -\frac{2\beta_m \beta_n}{a^2 J_0(x_n) J_0(x_m)} \int_0^a \frac{d}{dr} \{r^2 J_1(\beta_m r) J_1(\beta_n r)\} dr = 0. \quad \text{.....(A,03)}$$

In these reductions we utilize some properties of Bessel functions, namely $J_1(z) = -J_0'(z)$, and $\frac{d}{dz} \{z J_0'(z)\} = -z J_0(z)$

Solving (A,02) and (A,03) as two simultaneous equations determining α_{nm} and α_{mn} , we obtain

$$\alpha_{nm} = \frac{2x_n^2}{x_n^2 - x_m^2}, \quad \alpha_{mn} = -\frac{2x_m^2}{x_n^2 - x_m^2}. \quad \text{.....(A,04)}$$

Similarly, when n is not equal to zero,

$$\begin{aligned} \alpha_{nn} &= \frac{1}{\pi a^2 J_0(x_n)^2} \left[\int_0^a J_0(\beta_n r)^2 2\pi r dr + \int_0^a \beta_n r J_0'(\beta_n r) J_0(\beta_n r) 2\pi r dr \right] \\ &= \frac{1}{\pi a^2 J_0(x_n)^2} \left[\pi a^2 J_0(x_n)^2 + \int_0^a \frac{d}{dr} \{(\beta_n r)^2 J_1(\beta_n r)^2\} \frac{\pi}{\beta_n^2} dr \right] \\ &= 1, \quad n \neq 0 \end{aligned} \quad \text{.....(A,05)}$$

$$\alpha_{0n} = 0 \quad \text{.....(A,06)}$$

$$\alpha_{n0} = \frac{1}{\pi a^2 J_0(x_n)} \int_0^a \beta_n r J_0'(\beta_n r) 2\pi r dr = -\frac{2x_n^2 J_2(x_n)}{\beta_n^2 a^2 J_0(x_n)} = 2 \quad \text{.....(A,07)}$$

$$\alpha_{00} = 1 \quad \text{.....(A,08)}$$

owing to the following properties of Bessel functions,

$$J_0(x_n) + J_2(x_n) = \frac{2}{x_n} J_1(x_n) = 0, \quad \text{and} \quad \frac{d}{dz} \{z^2 J_2(z)\} = z^2 J_1(z).$$

ii) We next calculate C_{nm} as follows,

$$\begin{aligned} C_{nm} &= \int_A v_n \frac{\partial^2 v_m}{\partial z^2} dS = \frac{\partial}{\partial z} \int_A v_n \frac{\partial v_m}{\partial z} dS - \int_A \frac{\partial v_n}{\partial z} \cdot \frac{\partial v_m}{\partial z} dS - \int_C v_n \frac{\partial v_m}{\partial z} \tan \theta ds \\ &= -\frac{\partial}{\partial z} (\mu \alpha_{mn}) - \mu^2 \sum_{l=1}^{\infty} \alpha_{nl} \alpha_{ml} + 2\mu^2 = \mu^2 (2 - \sum_{l=1}^{\infty} \alpha_{nl} \alpha_{ml}) \end{aligned}$$

where

$$\begin{aligned} \sum_{l=1}^{\infty} \alpha_{nl} \alpha_{ml} &= 4 + \sum_{l=1}^{\infty} \frac{x_n^2 x_m^2}{(x_n^2 - x_l^2)(x_m^2 - x_l^2)} + \alpha_{nm} \alpha_{mn} + \alpha_{nm} \alpha_{mn} \\ &= 6 + 4 \sum_{l=1}^{\infty} \frac{x_n^2 x_m^2}{(x_n^2 - x_l^2)(x_m^2 - x_l^2)} = 2 + \frac{8x_n^2 x_m^2}{(x_n^2 - x_m^2)^2} \end{aligned}$$

owing to (B,05) of Appendix B.

Hence,

$$C_{nm} = -8\mu^2 \frac{x_n^2 x_m^2}{(x_n^2 - x_m^2)^2}, \quad n \neq m, \quad n \neq 0, \quad m \neq 0. \quad \text{.....(A,09)}$$

Similarly

$$C_{nn} = -\mu^2 \left[3 + 4 \sum_{l=1}^{\infty} \frac{x_n^4}{(x_n^2 - x_l^2)^2} \right] = -\mu^2 \left(1 + \frac{1}{3} x_n^2 \right), \quad n \neq 0 \quad \text{.....(A,10)}$$

owing to (B,04) of Appendix B.

And

$$C_{n0} = \int_A v_n \frac{\partial^2 v_0}{\partial z^2} dS = -\frac{\mu^2}{a\sqrt{\pi}} \int v_n dS = 0 \quad \text{.....(A,11)}$$

$$C_{0n} = \int_A v_0 \frac{\partial^2 v_n}{\partial z^2} dS = -\int_A \frac{\partial v_0}{\partial z} \frac{\partial v_n}{\partial z} dS - \int v_0 \frac{\partial^2 v_n}{\partial z^2} \tan \theta ds = 0 \quad \text{.....(A,12)}$$

$$C_{00} = \int_A v_0 \frac{\partial^2 v_0}{\partial z^2} dS = \mu^2 \quad \text{.....(A,13)}$$

iii) Lastly, we calculate D_{nm} as

$$D_{nm} = \int_C v_n \frac{\partial v_m}{\partial z} \tan \theta \, ds = \mu a \cdot \frac{1}{a\sqrt{\pi}} \cdot \frac{-\mu}{a\sqrt{\pi}} \cdot 2\pi a = -2\mu^2 \quad \dots\dots\dots (A,14)$$

$$\text{Similarly} \quad D_{nn} = 2\mu^2 \quad \dots\dots\dots (A,15)$$

$$\text{and} \quad D_{n0} = D_{0n} = D_{00} = -2\mu^2 \quad \dots\dots\dots (A,16)$$

Appendix B

i) We obtain from (3,12) and (3,13)

$$\frac{\beta_n r J_1(\beta_n r)}{J_0(x_n)} = -2 \left[1 + x_n^2 \sum_{m=1}^{\infty} \frac{1}{x_n^2 - x_m^2} \frac{J_0(\beta_m r)}{J_0(x_m)} \right], \quad \dots\dots\dots (B,01)$$

and putting in this equation $r=a$

$$\sum_{m=1}^{\infty} \frac{1}{x_n^2 - x_m^2} = -\frac{1}{x_n^2}. \quad \dots\dots\dots (B,02)$$

ii) Then, multiplying both sides of (B,01) by $\beta_n r dr$ and integrating from 0 to r and utilizing (B,01) (B,02) and (B,03), we obtain

$$\begin{aligned} \frac{(\beta_n r)^2 J_2(\beta_n r)}{J_0(x_n)} &= -2 \left[\frac{(\beta_n r)^2}{2} + x_n^4 \sum_{m=1}^{\infty} \frac{\beta_m r}{x_m^2 (x_n^2 - x_m^2)} \frac{J_1(\beta_m r)}{J_0(x_m)} \right] \\ &= -(\beta_n r)^2 + \frac{1}{2} x_n^2 - 8 + 4x_n^4 \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(x_n^2 - x_m^2)(x_m^2 - x_l^2)} \frac{J_0(\beta_l r)}{J_0(x_l)} \end{aligned}$$

and putting in this equation $r=a$,

$$\sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(x_n^2 - x_m^2)(x_m^2 - x_l^2)} = \frac{2}{x_n^4} - \frac{1}{8} \frac{1}{x_n^2} \quad \dots\dots\dots (B,03)$$

The left side of (B,03) is rewritten by changing the order of summation in the form,

$$\frac{1}{x_n^2} \sum_{l=1}^{\infty} \frac{1}{x_l^2} - 3 \sum_{m=1}^{\infty} \frac{1}{(x_n^2 - x_m^2)^2} = \frac{1}{8} \frac{1}{x_n^2} - \frac{1}{x_n^4} - 3 \sum_{m=1}^{\infty} \frac{1}{(x_n^2 - x_m^2)^2}$$

and therefore we obtain

$$\sum_{m=1}^{\infty} \frac{1}{(x_n^2 - x_m^2)^2} = \frac{1}{12} \frac{1}{x_n^2} - \frac{1}{x_n^4}. \quad \dots\dots\dots (B,04)$$

iii) Next two relations are necessary for computation of C_{nm} etc.

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{1}{(x_n^2 - x_l^2)(x_m^2 - x_l^2)} &= \sum_{l=1}^{\infty} \frac{1}{x_n^2 - x_m^2} \left(\frac{1}{x_m^2 - x_l^2} - \frac{1}{x_n^2 - x_l^2} \right) \\ &= \frac{2}{(x_n^2 - x_m^2)^2} - \frac{1}{x_n^2 x_m^2}, \quad \dots\dots\dots (B,05) \end{aligned}$$

$$\begin{aligned} \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{x_m^4 (x_n^2 - x_m^2)} &= \sum_{m=1}^{\infty} \frac{1}{x_n^2} \frac{1}{x_m^2} \left(\frac{1}{x_m^2} + \frac{1}{x_n^2 - x_m^2} \right) = \frac{1}{192} \frac{1}{x_n^2} - \frac{1}{x_n^6} + \sum_{m=1}^{\infty} \frac{1}{x_n^4} \left(\frac{1}{x_m^2} + \frac{1}{x_n^2 - x_m^2} \right) \\ &= \frac{1}{192} \frac{1}{x_n^2} + \frac{1}{8} \frac{1}{x_n^4} - \frac{3}{x_n^6} \quad \dots\dots\dots (B,06) \end{aligned}$$

owing to (B,08) and (B,09).

iv) And r^2 can be expressed by the eigen-function expansion of v_n as

$$r^2 = \frac{1}{2} a^2 + 4a^2 \sum_{m=1}^{\infty} \frac{1}{x_m^2} \frac{J_0(\beta_m r)}{J_0(x_m)} \quad \dots\dots\dots (B,07)$$

because of the integration

$$\begin{aligned} \int_0^a r^2 v_m 2\pi r dr &= \frac{1}{a\sqrt{\pi} J_0(x_n)} \int_0^a r^2 J_0(\beta_m r) 2\pi r dr = \frac{2\pi}{a\sqrt{\pi} \beta_m^4 J_0(x_m)} \int_0^{x_m} z^3 J_0(z) dz \\ &= \frac{-4\pi}{a\sqrt{\pi} \beta_m^4 J_0(x_n)} \int_0^{x_n} z^2 J_1(z) dz = -\frac{4\pi x_m^2 J_2(x_m)}{a\sqrt{\pi} \beta_m^4 J_0(x_m)} = \frac{4a\sqrt{\pi}}{\beta_m^2}. \end{aligned}$$

From (B,07), putting $r=a$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{x_n^2} = \frac{1}{8} \quad \dots\dots\dots (B,08)$$

iv) Multiplying both sides of (B,07) by $4rdr$ and integrating from 0 to r , we obtain

$$\begin{aligned} r^4 &= 4 \int_0^r r^3 dr = 4 \int_0^r \left[\frac{1}{2} + \sum_{m=1}^{\infty} \frac{4}{x_m^2} \frac{J_0(\beta_m r)}{J_0(x_m)} \right] a^2 r dr \\ &= a^2 r^2 + 16a^4 \sum_{m=1}^{\infty} \frac{1}{x_m^4} \cdot \frac{\beta_m r J_1(\beta_m r)}{J_0(x_m)} \end{aligned}$$

and utilizing (B,01) (B,02) and (B,07)

$$r^4 = 2a^2 r^2 - \frac{1}{2} a^4 - 32a^4 \sum_{m=1}^{\infty} \frac{1}{x_m^4} - 64 \sum_{l=1}^{\infty} \frac{1}{x_l^4} \frac{J_0(\beta_l r)}{J_0(x_l)},$$

and putting in this equation, $r=a$

$$\sum_{n=1}^{\infty} \frac{1}{x_n^4} = \frac{1}{192} \quad \dots\dots\dots (B,09)$$

and lastly $r^4 = -\frac{2}{3} a^4 + 2a^2 r^2 - a^4 \sum_{n=1}^{\infty} \frac{64}{x_n^4} \cdot \frac{J_0(\beta_n r)}{J_0(x_n)} \quad \dots\dots\dots (B,10)$

vi) Quite similarly to v), we obtain from (B,10)

$$\begin{aligned} r^6 &= 6 \int_0^r r^5 dr = 6 \int_0^r \left[-\frac{2}{3} a^4 + 2a^2 r^2 - a^4 \sum_{n=1}^{\infty} \frac{64}{x_n^4} \frac{J_0(\beta_n r)}{J_0(x_n)} \right] r dr \\ &= -2a^4 r^2 + 3a^2 r^4 - 384a^6 \sum_{n=1}^{\infty} \frac{1}{x_n^6} \cdot \frac{\beta_n r J_1(\beta_n r)}{J_0(x_n)}. \end{aligned}$$

and utilizing (B,01) (B,02) (B,06) (B,07) (B,08) (B,09) and (B,10)

$$r^6 = \frac{3}{2} a^6 - 6a^4 r^2 + \frac{9}{2} a^2 r^4 + 768a^6 \sum_{n=1}^{\infty} \frac{1}{x_n^6} + 2304 a^6 \sum_{n=1}^{\infty} \frac{1}{x_n^6} \frac{J_0(\beta_n r)}{J_0(x_n)},$$

and putting in this equation, $r=a$

$$\sum_{n=1}^{\infty} \frac{1}{x_n^6} = \frac{1}{3072}, \quad \dots\dots\dots (B,11)$$

and lastly

$$r^6 = \frac{7}{4} a^6 - 6a^4 r^2 + \frac{9}{2} a^2 r^4 + 2304a^6 \sum_{n=1}^{\infty} \frac{1}{x_n^6} \frac{J_0(\beta_n r)}{J_0(x_n)}. \quad \dots\dots\dots (B,12)$$

References and Notes

- 1) Beranek, L. L., Acoustics, Mc Graw Hill Co., 1954, pp. 268.
- 2) Stevenson, A. F., "Exact and approximate equations for wave propagation in acoustic horns", J. appl. Phys., 22, 12, p 1461~1463, Dec. 1951
- 3) ———, "General theory of electromagnetic horns", J. appl. Phys., 22, 12, p. 1447~1460, Dec. 1951.
- 4) Maezawa, S., adress at the 34-th annual meeting of the Japan Society of Mechanical Engineers, Apr. 1957.
- 5) ———, adress at the 7th Japan National Congress for Applied Mechanics, Sept. 1957.
- 6) Stevenson used the term principal mode with reference to uniform tube, here it is used with reference to the horn of variable cross-section itself. Principal mode in this sense is composed of infinite numbers of coupled modes in Stevenson's sense.