5. APPLICATIONS

The results of the previous sections will be illustrated here by obtaining the wave equations and outgoing wave admittance relations for conical, exponential, and Bessel horns in the reference shape with a straight axis and circular cross section. The hyperbolic horn will be treated in somewhat greater detail.

The use of the synthesis relations will be illustrated by investigating what other horns have the admittance characteristics of the conical horn. A new family of horns obtained by further use of the synthesis relations will be treated in a subsequent paper.

To begin consider the conical horn of Fig. 3. In the reference shape the diameter is expressed as a function of x, the distance from the vertex, by $d=d_t(x/x_0)$ where x_0 is the vertex to throat distance; hence $\rho=\alpha$. Since ρ'' is zero, the wave equation is $F''+\mu^2F=0$, whence $A=1$, $\theta=\mu\alpha$, $q=\mu$, $\beta=1+(1/j\mu\alpha)$; and finally at the throat where $\alpha=1$, $\beta_t=1+(1/j\mu)$. The equivalent circuit at the throat is as shown in Fig. 4, in which the elements are independent of frequency and S_t is in square inches. The frequency f_0 is the reference frequency $c/2\pi x_0=2155/x_0$, x_0 in inches.

For the exponential horn in Fig. 5, define $d=d_t \exp(x/x_0)$ where x_0 fixes the scale of axial length; at $x=x_0$, $d=d_t \exp(1)=(2.718\cdots)d_t$. Thus $\rho=\exp\alpha$ and since $\rho=\rho'=\rho''$, the wave equation is $F''+(\mu^2-1)F=0$ for which the solution is $A=1$, $q=(\mu^2-1)^{\frac{1}{2}}$. Hence $\beta=[1-(1/\mu^2)]^{\frac{1}{2}}+(1/j\mu)=\beta_t$, and the mechanical circuit at the throat ($\alpha=0$) is as in Fig. 6. Note the variation of the resistive element with frequency; by comparing this expression with those from electrical filter theory, it is seen that this variation is that of the mid-shunt image impedance of a constant-K high-pass filter. This behavior permits the exponential horn to maintain its input resistance constant over a greater range than does the conical horn of same length and terminal diameters.

From the expression for β it is seen that the admittance is independent of the position of the throat, except for the scale factor of area. Thus the infinite exponential horn may be cut off at any section without disturbing the frequency dependence of the throat admittance. In horn loudspeaker development work this is useful since it permits adjustment of the throat size and therefore the load on the electromechanical motor with little change in the frequency characteristics of the load. This corresponds to the action of an ideal transformer.

The reference frequency f_0 is also the cut-off frequency, since below $\mu=1$ the coefficient of $F(K^2)$ becomes negative. In this region of non-transmission the circuit of Fig. 6 must be modified to take account of the absence of the real part. This leads to a single mass-like element of mechanical reactance $267S_t[(1/\mu)-((1/\mu^2)-1)^{\frac{1}{2}}]$, S_t in inches.

Of some theoretical importance are the Bessel horns[11] defined by $S=S_t(x/x_0)^n$ or $\rho=\alpha^{n/2}$. Thus the F equation is

$$F''+[\mu^2-(n(n-2)/4\alpha^2)]F=0,$$

whose solution is

$$F=\alpha^{\frac{1}{2}}[J_{(n-1)/2}(\mu\alpha)+j(-1)^nN_{(n-1)/2}(\mu\alpha)],$$

in which the J and N are the two independent solutions of Bessel's equation. From (3.5) the admittance may be evaluated, yielding the expressions obtained by Ballantine in the reference just cited. The significance of the Bessel horns lies in the fact that they bridge the gap between the conical ($n=2$) and exponential types. If various Bessel horns are fitted to the same throat and mouth diameters and length, then it

[11] S. Ballantine, J. Frank. Inst. 203, 85 (1927).

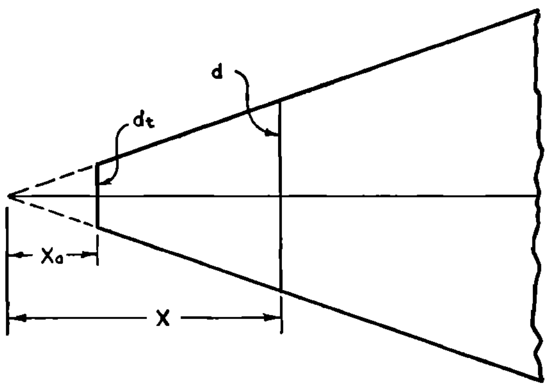

FIG. 3. Throat portion of infinite conical horn.

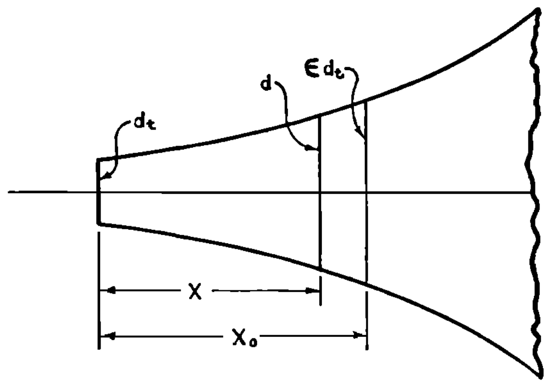

FIG. 5. Throat portion of infinite exponential horn. $\epsilon=2.718\cdots$.

has been shown[12] that as n increases without limit the shape and admittance characteristics of the exponential horn are approached.

As an example of the application of the plane wave analysis presented in Sections 2 and 3, let us consider the behavior of the hyperbolic horn reported by Freehafer.[7] The contour of the reference horn is given by $(d/d_t)^2-(x/x_0)^2=1$, where $d/2$ is Freehafer's ξ and x his ζ. See Fig. 7; note that when $x=x_0$, $d=(2)^{\frac{1}{2}}d_t$. Thus $\rho=(1+\alpha^2)^{\frac{1}{2}}$, and the F equation is

$$F''+[\mu^2-(1/(1+\alpha^2)^2)]F=0,$$

which is not soluble in terms of elementary functions. However, note that for $\mu<1/(1+\alpha^2)$, the coefficient of F changes sign and becomes negative. Thus at the throat, where $\alpha=0$, the region of the transmission lies above $\mu=1$. However, there is no sharp cut-off, since the coefficient is a function of α. Since ρ' is zero at the throat, the susceptance is $\sigma=-(1/\mu)(-A'/A)$. Now it is probable that the conductance will be small since it depends on q, the rate of change of phase, which decreases rapidly in the region of non-transmission. The net result is that the equivalent series resistance will probably have a maximum near $\mu=1$, falling rapidly below that frequency.

To obtain a more quantitative picture of the behavior of the hyperbolic horn, numerical integration (as described in the appendix) was employed, taking $\mu^2=0.25$, 0.50, 1.00, and 2.00. The integration was begun at a value of α such that the relation $1/(1+\alpha^2)^2<.005\mu^2$ was fulfilled; and the interval $\delta\alpha$ of integration was such as to ensure about a hundred points per "wave-length." It should be mentioned that these limits are much too stringent in view of the approximate nature of the fundamental assumptions; a more rational set would be starting at $1/(1+\alpha^2)^2 <.05\mu^2$, and using about 20 points per "wave-length."

The function $F=u-jv$ for starting the integration was obtained from the sinusoidal solution for $F''+\mu^2F=0$ at the initial values of α well removed from the throat. The integration was carried one step past the origin in order to be able to calculate $u'(0)$ and $v'(0)$. The throat admittance was then obtained by substituting

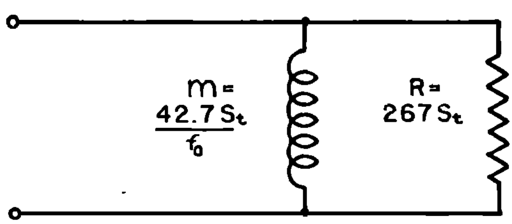

FIG. 4. Equivalent circuit at throat of infinite conical horn. The units are grams, square inches, cycles per second, and mechanical ohms.

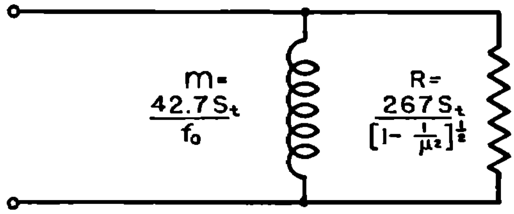

FIG. 6. Equivalent circuit at throat of infinite exponential horn. The units are grams, square inches, cycles per second, and mechanical ohms.

[12] C. R. Hanna, J. Frank. Inst. 203, 849 (1927).

the appropriate functions into Eq. (3.15). In order to compare the results with Freehafer's, the admittance was first inverted to the series impedance, which is R_s+jX_s in mechanical units. Then the abscissa was transformed from Freehafer's $2\pi a/\lambda=\pi d_t/\lambda=\mu \tan \theta_0$, θ_0 being the half-angle between the asymptotes to the hyperbolic contour. The resulting comparison is shown in Figs. 8 and 9, the curves being those calculated by Freehafer;[7] the coordinates are his. It is seen that the agreement is only fair, indicating that the numerical integration need not be too detailed. The plane wave equation for the hyperbolic horn differs but little from that for Freehafer's first radial function; hence the discrepancy is due almost entirely to the variation of sound pressure transverse to the axis of the horn, which in the exact theory introduces functions of θ_0 into the impedance.

Another comparison between the plane wave theory and the experimental results is afforded by the variation of axial sound pressure along the horn as given by Eq. (3.21) for the resultant pressure when both transmitted and reflected waves are present. Figures 10 and 11 show the experimentally determined[13] points for the 15° hyperbolic horn, compared to the curves as calculated using Eqs. (3.23) and (3.25). Other reference dimensions for the horn were $d_t=3.06''$ and $x_0=5.70''$; from the latter $f_0=378$ c.p.s. Note that the area factor in the pressure has been removed by plotting not P but the product $P(1+\alpha^2)^{\frac{1}{2}}$, thus yielding the absolute value of F for the essential variation. The constants ψ and φ were determined from these amplitude-axial distance curves by choosing the pair of values causing the experimental and calculated points to agree near the first maximum in the curve. This same pair of values was then used for the phase-axial distance curves, affording an independent check. It is seen that the agreement is good for the frequencies used; at higher values of the frequency parameter $\mu=kx_0=kd_t/2 \tan \theta_0$ it was found that the surfaces of constant phase in the horn were still regular, but the pressure amplitude distribution displayed such complexity that the axial pressure was no longer representative of that across each section.

[13] V. Salmon, Massachusetts Institute of Technology Physics Ph.D. Thesis (December, 1938).

As a last example consider the following problem in the synthesis of a horn. Using the conical horn admittance components, what other horns have similar conductance or susceptance functions?

First, given $\sigma=-1/\mu\alpha$. Insert this into (4.3), yielding for the conductance $\gamma'+\mu^2\alpha\gamma(1-\gamma^2)=0$. It is easily seen that $\gamma=1$ is one solution, corresponding to the conical horn. By a quadrature there is also obtained $\gamma=1/[1+D \exp (\mu\alpha)^2]$ where $D=D(\mu)$ is the constant of integration. When this is substituted into the shape-conductance Eq. (4.5) it will be found that only for $D=0$ is the coefficient of ρ independent of μ. This again yields $\gamma=1$ as the only solution permitting a realizable horn.

To work the other way, insert the conductance $\gamma=1$ into Eq. (4.3), leaving $(\sigma'/\sigma)-\mu\sigma=0$, whose solution is $\sigma=-1/[\mu\alpha+E(\mu)]$. Now, at zero frequency no wave energy is being transmitted so all of the impedance is reactive. For useful horns this reactance is always mass-like, corresponding to infinite negative susceptance at zero frequency. Hence $E(\mu)$, the integration constant, must be zero, yielding the conical horn susceptance $-1/\mu\alpha$. Thus only the conical horn has admittance components in the forms given.

Many other admittances may be tested in the same manner. In a succeeding paper it will be shown that one practical form leads to a new family of horns having useful throat impedance characteristics.

Thus in this section the analysis and the synthesis relations have been applied to various

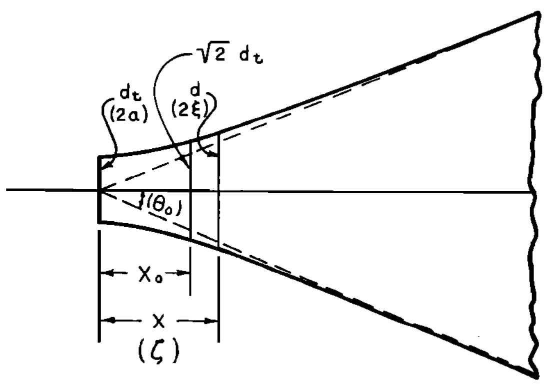

FIG. 7. Throat portion of infinite hyperbolic horn. The symbols in parentheses are those of Freehafer (reference 7).

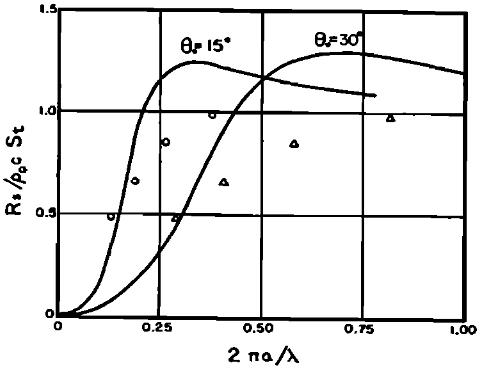

FIG. 8. Series resistance at throat of infinite hyperbolic horn. Curves from Freehafer (reference 7). Circles and triangles, plane wave calculations for 15° and 30° horns, respectively, from Eq. (3.15).

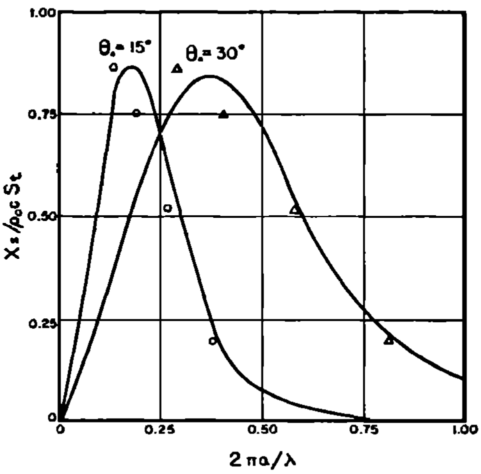

FIG. 9. Series reactance at throat of infinite hyperbolic horn. Curves from Freehafer (reference 7). Circles and triangles, plane wave calculations for 15° and 30° horns, respectively, from Eq. (3.15).

horns to illustrate the unifying action of the theory.

SUMMARY

In this paper plane wave horn theory has been generalized and unified so as to aid in quickly estimating the performance of a given horn, and to permit some degree of selection and synthesis of horns for a specified performance. Features of the analysis are the use of dimensionless variables, the selection of admittance as the pertinent behavior parameter, and the obtaining of relations among the admittance components and the shape parameter.

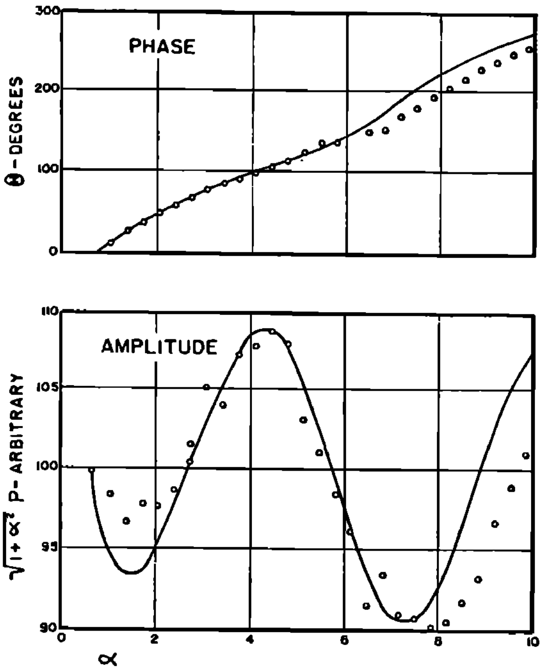

FIG. 10. Pressure $P \exp(-j\Theta)$ along axis of 15° finite hyperbolic horn. Throat diameter 3.06″, mouth diameter 38.1″, length 70.9″. Frequency parameter $\mu = f/f_0 = 191/378 = .505$. Circles by experiment, curves from Eqs. (3.23) and (3.25).

6. APPENDIX

Numerical Integration

It is desired to integrate Eq. (2.6) when two initial values are known. The method developed is based on the work of Hartree[14] and Manning and Millman.[15] The starting point is the expansion for the second finite central difference of a function in terms of the second derivative and its even finite second differences. This is

$$\delta^2 F = [F'' + (\delta^2 F''/12) + (\delta^4 F''/240) + \cdots](\delta\alpha)^2, \quad (6.1)$$

where δ^n is the nth order central difference and $\delta\alpha$ is the increment of the independent variable. Now if $\delta\alpha$ is chosen small enough the last term in the right-hand member will be small compared to the others, which may be satisfied by using at least 20 intervals per "wave-length" for the integration. In any event, it is easily tested by

[14] D. R. Hartree, Mem. and Proc. Manchester Lit. and Phil. Soc. 77, 91 (1933).

[15] M. F. Manning and J. Millman, Phys. Rev. 53, 673 (1938).

performing the differencing on the function used to get the initial values.

Using the approximation with two terms in the right-hand member we first substitute for $\delta^2 F$ the expression $F_{n-1}-2F_n+F_{n+1}$ obtained by differencing the sequence of functions F arranged in order at the equal intervals $\delta\alpha$; secondly, $\delta^2 F''$ is replaced by $F_{n-1}''-2F_n''+F_{n+1}''$; and finally for each F'' is substituted the appropriate $-K^2F$. This yields a relation in F_{n+1}, F_n, and F_{n-1} and produces the following approximate recursion formula for F_{n+1}:

$$F_{n+1}=\frac{F_n(2N-10K_n^2)-F_{n-1}(N+K_{n-1}^2)}{N+K_{n+1}^2}, \quad (6.2)$$

where

$$N\equiv 12/(\delta\alpha)^2.$$

For the integration the functions in parentheses may be computed beforehand for the whole range of integration. This offers no difficulties when K^2 is given analytically; but when the horn contour is known only as a table of values, it will be necessary to select a reasonable x_0 (as noted in Section 2) and to recast the table into ρ as a function of α. Then the ρ''/ρ are obtained for each α. In view of the assumptions involved in the plane wave theory, this could well be done graphically. A convenient sequence of columns for the actual integration is α, $2N-10K^2$, $N+K^2$, and F.

Since the first derivative is needed in the admittance relations, it is necessary to integrate one step past the throat position in order to use the following approximate expression for F'.

$$F_n'=\frac{F_{n+1}-F_{n-1}}{2\delta\alpha}+\frac{\delta\alpha}{12}[K_{n+1}^2F_{n+1}-K_{n-1}^2F_{n-1}]. \quad (6.3)$$

This is based on the relations given in Hartree.[14]

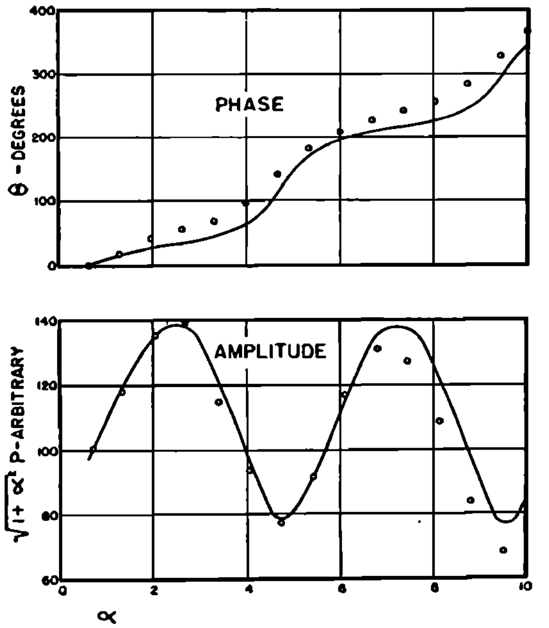

FIG. 11. Pressure $P\exp(-j\Theta)$ along axis of 15° finite hyperbolic horn. Frequency parameter $\mu=258/378=.682$. Dimensions as for Fig. 10. Circles by experiment, curves from Eqs. (3.23) and (3.25).