

where ν is the frequency of the wave. The differential equation for ψ (obtained by putting ν into the wave equation) is

$$\nabla^2 \psi + \frac{4\pi^2 \nu^2}{v^2} \psi = 0$$

where v is the velocity of the wave. Let $K = \frac{2\pi}{\lambda}$ where λ = wavelength.

Equation (1) in oblate spheroidal coordinates is

$$\frac{1}{\cosh \mu} \frac{\partial}{\partial \mu} (\cosh \mu \frac{\partial \psi}{\partial \mu}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \psi}{\partial \theta}) + \frac{\cosh^2 \mu - \sin^2 \theta}{\cosh^2 \mu \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{K^2 \alpha^2 (\sinh^2 \mu + \cos^2 \theta)}{4} \psi = 0 \quad (1a)$$

Put $\psi = M(\mu) \Theta(\theta) \Phi(\varphi)$

Equation (1a) separates in a straightforward way, yielding the three ordinary differential equations

$$\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -\alpha^2 \quad (2)$$

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\Theta}{d\theta}) - \frac{\alpha^2}{\sin^2 \theta} + \frac{\alpha^2 K^2}{4} \cos^2 \theta = \beta^2 \quad (3)$$

$$\frac{1}{M \cosh \mu} \frac{d}{d\mu} (\cosh \mu \frac{dM}{d\mu}) + \frac{\alpha^2}{\cosh^2 \mu} + \frac{\alpha^2 K^2}{4} \sinh^2 \mu = \gamma^2 \quad (4)$$

α^2 and β^2 are the separation constants. ψ is single-valued and hence α must be an integer.

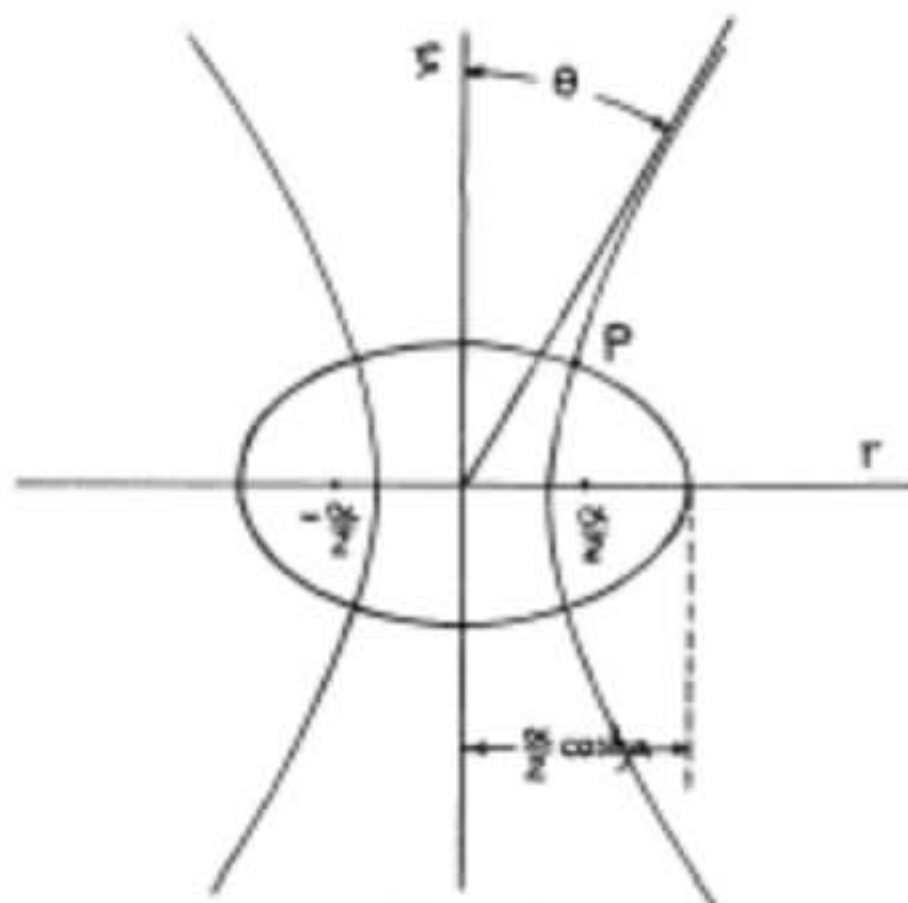


Fig. 1

Now let us consider equation (3). We shall first change the dependent variable, defining W by the equation

$$\Theta = W \sin^a \theta$$

Equation (3) becomes

$$\frac{d^2 W}{d\theta^2} + \frac{2a+1}{\sin \theta} \cos \theta \frac{dW}{d\theta} + (-a^2 - a + \frac{\kappa^2 \alpha^2}{4} \cos^2 \theta - \beta^2) W = 0$$

We now change independent variables by putting

$$z = \cos \theta$$

The equation for W is

$$(1-z^2) \frac{d^2 W}{dz^2} - 2(a+1)z \frac{dW}{dz} + (-a^2 - a - \beta^2 + \frac{\kappa^2 \alpha^2}{4} z^2) W = 0$$

or

$$(1-z^2) \frac{d^2 W}{dz^2} - 2(a+1)z \frac{dW}{dz} + (b - c^2 z^2) W = 0 \quad (5)$$

where

$$b = (-a^2 - a - \beta^2)$$

$$c^2 = -\frac{\kappa^2 \alpha^2}{4}$$

Putting in the value of κ we find

$$c^2 = -\frac{\left(\frac{2\pi}{\lambda}\right)^2 \left(\frac{\alpha}{2} \sin \theta_0\right)^2}{\sin^4 \theta_0} = -\left(\frac{2\pi}{\sin \theta_0}\right)^2 \left(\frac{\alpha}{2} \frac{\sin \theta_0}{\lambda}\right)^2$$

Radiation of shorter wavelength than the radius of the throat is not appreciably diffracted; accordingly, the maximum numerical value for c^2 in which we are interested is given by

$$|c^2_{\text{Max}}| = \left(\frac{2\pi}{\sin \theta_0}\right)^2$$

For a 15° horn $-600 < c^2 < 0$

For a 30° horn $-160 < c^2 < 0$

Let us now return to equation (4), and change the dependent variable to F and the independent variable to x where

$$M = F \cosh^a \mu$$

$$x = \sinh \mu$$

Equation (4) then becomes

$$(1+x^2) \frac{d^2 F}{dx^2} + 2(a+1)x \frac{dF}{dx} + (a^2 + a + \beta^2 + \frac{\kappa^2 \alpha^2}{4} x^2) F = 0$$

or using the notation introduced previously

$$(1+x^2) \frac{d^2 F}{dx^2} + 2(a+1)x \frac{dF}{dx} - (b + c^2 x^2) F = 0 \quad (6)$$

It is worth noting that by putting $x = iz$, equation (6) is reduced to equation (5). With the equations written in the form of (5) and (6) both x and z are real and $0 < x$ while $z_0 < z < 1$.

Now we shall seek V , a complex solution of the wave equation whose real part is the real velocity potential within the region bounded by the surfaces $\theta = \theta_0$, $\mu = 0$ and $\mu = \infty$. We shall call this region R . Then V must satisfy the following conditions:

- (1) It must be finite and single-valued in R .
- (2) Its normal derivative must vanish over the surface $\theta = \theta_0$.
- (3) Its normal derivative at $\mu = 0$ must be a function $u(\theta, \varphi, t)$ prescribed by the behavior of the diaphragm.
- (4) For large values of μ , it must represent waves traveling out from the origin.

Let us consider the function u . If it is to represent conditions at the diaphragm which make sense physically, it must be a periodic function of φ with period 2π . We shall assume that it is also periodic in t with period 2π . Now condition (2) requires that $d\theta/ds = 0$. We can satisfy this requirement by choosing b so that for a given pair of values of a and c , $d\theta/ds = 0$. From this fact and the differential equation for the θ 's, it can be shown in the standard way that the θ 's are orthogonal. That is,

$$\int_0^{\theta_0} \sin \theta \, \Theta_{abc} \, \Theta_{a'bc} \, d\theta = 0 \quad \text{when } b \text{ and } s \text{ are unequal.}$$

According to the Sturm-Liouville theory the set of θ 's is also complete, so that we are able to write.

$$u(\theta, \varphi, t) = \sum_{a,b,c} B_{abc} \Theta_{abc} e^{2\pi i a \varphi} e^{i \omega_c t}$$

In the notation used above, a , b or c written as a subscript is intended to represent the ordinal number decreased by one¹ of the particular member of the set of permissible values which the parameter is assuming at the time.

¹ We use b_n to represent the $(n+1)$ st characteristic value because the $(n+1)$ st characteristic function has n zeros.