

**Effects of bending portions of the air column on the acoustical resonances of a
wind instrument**

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Abstract

The need to keep long wind musical instruments compact imposes the bending of portions of the air column. Although manufacturers and players mention its effects as being significant, the curvature is generally not included in physical models and only a few studies, in only simplified cases, attempted to evaluate its influence. The aim of the study is to quantify the influence of the curvature both theoretically and experimentally. A multimodal formulation of the wave propagation in bent ducts is used to calculate the resonances frequencies and input impedance of a duct segment with a bent portion. From these quantities an effective length is defined. Its dependence on frequency is such that, compared to an equivalent straight tube, the shift in resonance frequencies in a tube with bent sections is not always positive, as generally stated. The curvature does not always increase the resonances frequencies, but may decrease them, resulting in a complex inharmonicity. An experimental measurement of the effect of the curvature is also shown, with good agreement with theoretical predictions.

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I. INTRODUCTION

Long wind musical instruments are often made with bends to keep them compact. Although it is generally stated, following Rayleigh's argument, that "*straightness is not a matter of importance*,"¹ bending portions of the air column may affect the tuning in a musically significant way. Several authors attempted to quantify the effect of the curvature on the tuning. Nederveen^{2,3} and later Keefe and Benade⁴ showed that in a bent duct, in the long wavelength limit, the inertance is lowered relative to its value in a straight duct having the same cross-section. The curvature induces an increase in resonance frequencies. In other words, the length correction due to the curvature in a duct is negative. Experimental studies reported in the literature³⁻⁵ confirm this qualitative result at low frequency. Following these studies, methods for compensating bends in air columns when designing wind instruments have been proposed, as, for example, a reduction of the bore diameter at bends, combined with a lengthening when the frequency is to be the same as the straight duct of the original length.^{2,3,5,6}

However, reducing the bore diameter at bends to compensate for the tuning change supposes that this change is weakly frequency-dependent, or, at least, that the frequency shift, or the length correction, remains the same sign over the musically useful frequency range.

It is shown in this paper that the frequency shift in a bend may vary significantly over frequency and that its sign changes. The curvature does not always increase the resonance frequencies, but may decrease them, resulting in a complex inharmonicity. Two ways of characterizing the effect of the curvature are described in the following: the direct computation of the resonance frequencies in a duct with a bend, and the computation of its input impedance. Both quantities are calculated using several theoretical or numerical methods

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that are first described to detail all the necessary quantities and notations (Sec. II). Then results are given and discussed (Sec. III), followed by an experimental verification (Sec. IV).

II. METHODS

A. General formulation

For simplicity and in order to clearly point out the effect of bending, the air column is supposed to have a constant, circular, cross-section. Since any such waveguide can be represented by a succession of cylindrical and toroidal segments, we will consider in the following the duct drawn in Fig. 1. The curvature of the middle segment is quantified by the adimensional axis curvature $\kappa = a/R_0$, where R_0 is the axis radius of curvature.

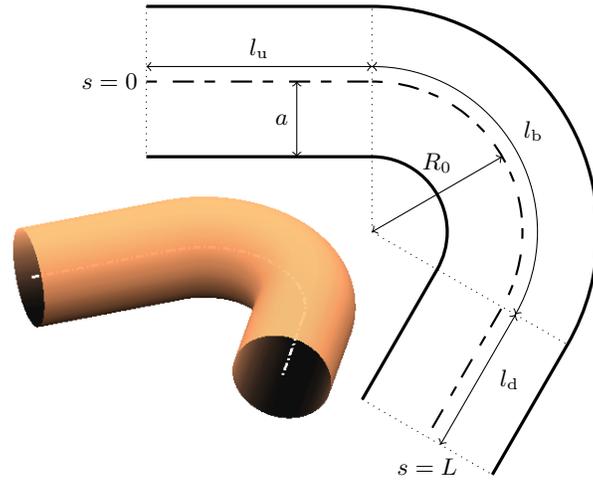


FIG. 1. Duct with a bent portion. s is the abscissa along the waveguide axis, measured from the input of the duct (top), and $L = l_u + l_b + l_d$ is the total length of the duct axis.

The waveguide is filled with an inviscid and homogeneous medium, with density ρ_0 and sound velocity c_0 (in Sec. IV, approximations of the viscothermal losses are introduced to allow the comparison with experimental results). Under the assumption of linear propagation, the acoustic pressure and axial velocity in the waveguide are the solutions of the coupled

equations [the time dependence $\exp(-j\omega t)$ is omitted]

$$\frac{\partial}{\partial s} \begin{pmatrix} p \\ Av_s \end{pmatrix} = \begin{pmatrix} 0 & jkZ_c h_s \\ -\frac{h_s}{jkZ_c} (\Delta_{\perp}^{(\kappa)} + k^2) & 0 \end{pmatrix} \begin{pmatrix} p \\ Av_s \end{pmatrix}, \quad (1)$$

where $A = \pi a^2$ is the cross-section area, $Z_c = \rho_0 c_0 / A$ is the characteristic impedance, and $k = \omega / c_0$ is the wavenumber. In the bent segment, $\Delta_{\perp}^{(\kappa)}$ is the transverse Laplacian in the toroidal coordinates (s, r, θ) , with s the abscissa along the waveguide axis and (r, θ) the polar coordinates in the transverse cross-section:^{7,8}

$$\Delta_{\perp}^{(\kappa)} = \frac{1}{rh_s} \left[\frac{\partial}{\partial r} \left(rh_s \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{h_s}{r} \frac{\partial}{\partial \theta} \right) \right], \quad (2)$$

with $h_s = 1 - \kappa r \cos \theta / a$. In the cylindrical segments, $\kappa = 0$ and $\Delta_{\perp}^{(0)}$ is the transverse Laplacian in cylindrical coordinates:

$$\Delta_{\perp}^{(0)} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \quad (3)$$

The resonances of the duct combination can be studied in two ways:

1. by a direct calculation of the resonance frequencies,
2. or by calculating the impedance at the input of the duct.

1. Resonance frequencies

The general principle for calculating the resonance frequencies is the following. From the coupled equations (1), one deduces a relation between the acoustic fields at each end of the duct. For example, the pressure and axial velocity field at the output ($s = L$) are written as function of the fields at the input ($s = 0$):

$$(p(L, r, \theta), v_s(L, r, \theta)) = H(p(0, r, \theta), v_s(0, r, \theta)), \quad (4)$$

with H an operator and frequency dependent. Then, given this relation and the boundary conditions at each end of the duct, a dispersion relation is written, the solutions of which

are the resonance frequencies $f_i^{(\kappa)}$ ($i = 1, 2, 3, \dots$) in the duct combination. Practically, this problem is solved by discretization of the solutions (cf. § II.C) or the differential operators (see § II.D) in Eq. (1), so that Eq. (4) and the dispersion relation become linear matrix equations.

In the following, the curvature induced shift of the resonance frequencies will be expressed as a correction to the length L of the duct:

$$\Delta L_i = \left(\frac{f_i^{(0)}}{f_i^{(\kappa)}} - 1 \right) L, \quad (5)$$

where the $f_i^{(0)}$ are the resonance frequencies in a cylindrical duct having the same radius a and length L as the duct, with the same boundary conditions imposed at the ends. For simple end conditions, the $f_i^{(0)}$ are known analytically. For example, if one imposes at the input end ($s = 0$) a Neumann condition $\partial p / \partial s = 0$ and, at the output end ($s = L$), a Dirichlet condition $p = 0$ as the simplest, non radiating, open end condition, then the resonance frequencies of this "closed-open" duct are

$$f_i^{(0)} = \left(i - \frac{1}{2} \right) \frac{c_0}{2L}, \quad i = 1, 2, 3, \dots \quad (6)$$

for frequencies under the cutoff of the first superior mode in the duct, that is, $f_c = \gamma_{10} c_0 / 2\pi a$, with $\gamma_{10} \simeq 1.8412$ the first positive zero of the first derivative of the Bessel function of the first kind J_1 .

2. Input impedance

Classically, the resonator composing of a wind instrument is characterized by calculating or measuring its input impedance

$$Z_{\text{in}}^{(\kappa)} = \bar{p}(0) / A\bar{v}_s(0) \quad (7)$$

as a function of the frequency, where $\bar{p}(0)$ and $A\bar{v}_s(0)$ are, respectively, the mean pressure and volume velocity at the input end of the duct. The impedance proves valuable to study

the linear acoustic properties of the instrument, since, in particular, its peaks are strongly linked to the playing frequencies of the instrument.

In a cylindrical duct, assuming that only plane waves can propagate ($f < f_c$), the input impedance reads as

$$Z_{\text{in}}^{(0)} = Z_c \tanh \left(-jkL + \operatorname{arctanh} \left[\frac{Z(L)}{Z_c} \right] \right), \quad (8)$$

with $Z(L)$ the value of the impedance at the output end of the duct. If the curvature of some region of the duct is not zero, then the input impedance cannot be written analytically. Therefore, approximate analytical solutions (cf. Sec. II.B) or more complex formulations (cf. Sec. II.C) are used.

Once the input impedance Z_{in} is computed, it can be recalculated into a length correction. Using Eq. (8), an effective length of the duct is defined:

$$L_{\text{eff.}} = \operatorname{Re} \left[-\frac{1}{jk} \left(\operatorname{arctanh} \left[\frac{Z_{\text{in}}^{(\kappa)}}{Z_c} \right] - \operatorname{arctanh} \left[\frac{Z(L)}{Z_c} \right] \right) \right], \quad (9)$$

leading to the length correction

$$\Delta L(f) = L_{\text{eff.}} - L. \quad (10)$$

Following are explained three methods to solve the general equations given above and characterize the effect of the curvature on the resonances of the duct.

B. Low frequency approximation

Considering the problem in the long wavelength limit, Nederveen^{2,3} assumes the pressure field in a bend to be constant over the circular cross-section: $p(s, r, \theta) = \bar{p}(s)$. Then, Eq. (1) can be simplified by removing the term involving $\Delta_{\perp}^{(\kappa)} p$, and the integration of the equation over the cross-section leads to

$$\frac{\partial}{\partial s} \begin{pmatrix} \bar{p} \\ A\bar{v}_s \end{pmatrix} = \begin{pmatrix} 0 & jk\alpha(\kappa)Z_c \\ jk/Z_c & 0 \end{pmatrix} \begin{pmatrix} \bar{p} \\ A\bar{v}_s \end{pmatrix}, \quad (11)$$

where

$$\alpha = \frac{\frac{1}{2}\kappa^2}{1 - \sqrt{1 - \kappa^2}} < 1. \quad (12)$$

Then, the effect of bending a duct is shown as a perturbation to the inertance, lowered relative to its value in a straight duct. By decomposing the toroidal waveguide into a set of parallel rectangular slices, Keefe and Benade⁴ propose in 1983 another expression of the inertance correction, that is shown to be analytically identical the Nederveen's formula.³

This low frequency approximation of the evolution equations for p and v_s is amenable to an analytical solution for the input impedance³ and, as it will be shown later in the paper, it gives a good approximation of the propagation constant of the first bend mode in the long wavelength limit. However, the assumption of a uniform pressure over the cross-section is a strong assumption, since, even below the cutoff frequency f_c , the variations of the pressure and velocity over the cross-section are not negligible.^{7,9} Therefore, this solution is limited to a very low frequency range and cannot account for the dependence on frequency of the effect of curvature.

To take into account the transverse variations of the pressure and velocity fields, higher order modes, even though they are evanescent, must be taken into account. This is the aim of the following method.

C. Multimodal method

The wave field in a straight duct is commonly written as a series decomposition on the basis of the transverse modes

$$\phi_n^{(0)}(r, \theta) = \Lambda_n J_\mu(\gamma_{\mu\nu} \frac{r}{a}) \sin\left(\mu\theta + \frac{\sigma\pi}{2}\right), \quad (13)$$

where Λ_n is a normalizing factor and $\gamma_{\mu\nu}$ is the $(\nu + 1)$ th zero of J'_μ , the derivative of the i th-order Bessel function of the first kind [the modes are sorted in increasing order of their cutoff frequency, such that n is a integer index, referring to the triplet (μ, ν, σ) of the usual circumferential, radial, and symmetry indexes.^{7,8} σ is either 0 or 1. $n = 0$ denotes the plane wave mode]. These modes are the well known eigenfunctions of the transverse Laplacian

$\Delta_{\perp}^{(0)}$ in the two-dimensional (2D) circular cross-section of the waveguide. In a bend, the transverse Laplacian $\Delta_{\perp}^{(\kappa)}$ is not separable in toroidal coordinates; therefore the bend modes $\phi_n^{(\kappa)}$ cannot be calculated analytically. However, since the basis $\{\phi_n^{(0)}\}_{n \geq 0}$ is a complete orthogonal basis of the circular cross-section, it can be used as a *local* transverse basis to exactly decompose the wave field:

$$p(s, r, \theta) = \sum_{n \geq 0} \phi_n^{(0)}(r, s) P_n(s), \quad (14)$$

$$Av_s(s, r, \theta) = \sum_{n \geq 0} \phi_n^{(0)}(r, s) U_n(s). \quad (15)$$

The projection of Eq. (1) on the basis $\{\phi_n^{(0)}\}$ gives the matrix coupled equations for the vectors \vec{P} and \vec{U} of the components P_n of pressure and U_n of volume velocity:

$$\frac{\partial}{\partial s} \begin{pmatrix} \vec{P} \\ \vec{U} \end{pmatrix} = \begin{pmatrix} 0 & jkZ_c(\mathcal{I} - \kappa\mathcal{A}) \\ \frac{1}{jkZ_c} \left[\mathcal{H}^2(\mathcal{I} - \kappa\mathcal{A}) + \frac{\kappa}{a^2} \mathcal{B} \right] & 0 \end{pmatrix} \begin{pmatrix} \vec{P} \\ \vec{U} \end{pmatrix}, \quad (16)$$

where \mathcal{I} is the identity matrix, \mathcal{H} is a diagonal matrix given by $\mathcal{H}_{nn}^{(0)} = jk_n^{(0)}$, $k_n^{(0)} = \sqrt{k^2 - (\gamma_{\mu\nu}/a)^2}$ the propagation constant of the mode $\phi_n^{(0)}$, and \mathcal{A} and \mathcal{B} are constant matrices, functions of $\{\phi_n^{(0)}\}$.

This formulation calls for several comments:

- Taking only one mode into account in Eq. (16) does not give Eq. (11). This is because projecting p and Av_s on the plane wave mode $\phi_0^{(0)}$ only is equivalent to assuming that both the pressure and axial velocity are uniform over the cross-section. Logically, no effect of the curvature arises at this order of the modal decomposition.
- Since the matrix coefficients in the system (16) are constant along the axis, the solutions for the wavefield, the scattering matrix or the impedance matrix (see below) can be obtained algebraically, as functions of the matrix^{7,8,10}

$$\mathcal{M} = (\mathcal{I} - \kappa\mathcal{A}) \left[\mathcal{H}^2(1 - \kappa\mathcal{A}) + \frac{\kappa}{a^2} \mathcal{B} \right]. \quad (17)$$

- In the case of a straight waveguide segment ($\kappa = 0$), the matrix \mathcal{M} reduces to \mathcal{H}^2 , then the solution of Eq. (16) is the classical modal decomposition mentioned above.
- The bend modes $\phi_n^{(\kappa)}$ appear as the basis that diagonalizes the matrix \mathcal{M} when an infinite number of modes is taken into account.^{9,11} It is thus possible to approximate these modes and their propagation constants $k_n^{(\kappa)}$ by computing the eigensolutions of the truncated matrix \mathcal{M} .

1. Resonance frequencies

In order to derive a dispersion relation and calculate the resonance frequencies in the duct, the vector field \vec{P} at the left and right ends of the segment is split into ingoing waves \vec{A} and outgoing waves \vec{B} . The scattering matrix relates outgoing waves to ingoing waves:

$$\begin{pmatrix} \vec{B}_{\text{left}} \\ \vec{B}_{\text{right}} \end{pmatrix} = \mathcal{S}(f) \begin{pmatrix} \vec{A}_{\text{left}} \\ \vec{A}_{\text{right}} \end{pmatrix}. \quad (18)$$

For the kinds of duct we are interested in (toroidal and cylindrical segments), \mathcal{S} can be algebraically calculated.⁷ Since the same basis $\{\phi_n^{(0)}\}$ is used for the modal decomposition in each segment, whatever the value of the curvature, writing the continuity at the interfaces is straightforward. The vector pressure and volume velocity fields $\vec{P}(s)$ and $\vec{U}(s)$ are continuous and, as a consequence, the scattering matrix of a set of connected segments can be explicitly written as function of the scattering matrices of each segment.⁷ Once the scattering matrix of the duct combination is calculated and given the boundary conditions, the dispersion relation is written. For example, for the "closed-open" waveguide, the boundary conditions are $\vec{B}_{\text{left}} = \vec{A}_{\text{left}}$ and $\vec{B}_{\text{right}} = -\vec{A}_{\text{right}}$, then

$$\det \left[\mathcal{S}(f) - \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix} \right] = 0, \quad (19)$$

where \mathcal{I} is the identity matrix and \det denotes the determinant. The resonance frequencies are the solution of Eq. (19).

In the case of a straight duct, using 12 modes $\phi_n^{(0)}$, the solutions of Eq. (19) differ from the theoretical results (Eq. 6) by less than 10^{-12} %, thus mainly limited by the numerical precision of the computer.

2. Input impedance

In the frame of the multimodal formalism, an impedance matrix can be defined, fulfilling

$$\vec{P}(s) = \mathcal{Z}(s)\vec{U}(s) \quad (20)$$

(throughout this paper, the classical scalar impedance is denoted with a roman capital Z , while a script letter is used to denote the impedance matrix, as well as all matrix variables). It is governed by a matrix Riccati equation,^{9,12,13} a solution of which can be written algebraically in the case of a bent or straight segment as studied in this paper. The impedance matrix is continuous at the interfaces between two segments.

Once the boundary condition at the output end is expressed as a matrix $\mathcal{Z}(L)$, the input impedance matrix $\mathcal{Z}^{(\text{in})}$ of the duct is calculated. As, by definition, $P_0 = \bar{p}$ and $U_0 = A\bar{v}_s$, we can write

$$\bar{p}^{(\text{in})} = \mathcal{Z}_{00}^{(\text{in})} A\bar{v}_s^{(\text{in})} + \sum_{n>0} \mathcal{Z}_{0n}^{(\text{in})} U_n. \quad (21)$$

Below the first cut-off and far from discontinuities, sources, or other regions where a mode coupling may occur, $U_{n>0} \rightarrow 0$. Thus, providing that l_u is large enough, the scalar input impedance z_{in} can be approximated by the top left value $\mathcal{Z}_{00}^{(\text{in})}$ of the impedance matrix:

$$Z_{\text{in}}^{(\kappa)} = \frac{\bar{p}^{(\text{in})}}{A\bar{v}_s^{(\text{in})}} \simeq \mathcal{Z}_{00}^{(\text{in})}. \quad (22)$$

In the case of a straight duct, using 12 modes $\phi_n^{(0)}$ to compute the impedance matrix, its corner value $\mathcal{Z}_{00}^{(\text{in})}$ differs from the analytical solution (8) by roughly 10^{-12} % over the frequency range of interest (from 0 to the cutoff f_c), certainly depending on the computer precision.

D. Finite difference method

Resonance frequencies of pipe combinations were independently computed numerically using the finite difference method (FDM). A discretization grid is fitted in the tube combination, as schematically shown in Fig. 2.

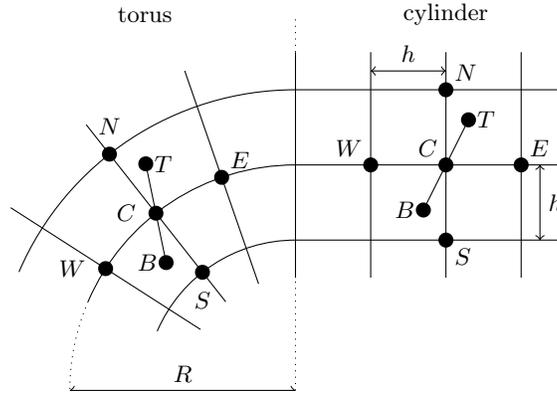


FIG. 2. Grid layout for the FDM solution in a bent segment connected to a straight segment.

The central point C is surrounded by points to the north (N), south (S), west (W), and east (E), and perpendicular to the plane through these points by top (T) and bottom (B) points³. In the cylindrical part the mesh is Cartesian with a mesh size h . In the toroidal part, R is the local radius of curvature, R_0 the radius of curvature of the center line at which radius the mesh size is equal to h , so the angle between two adjacent grid lines is h/R_0 .

The discretization equation for the Helmholtz equation in the toroidal part in cylindrical coordinates is

$$\left(1 + \frac{h}{2R}\right) \frac{p_N - p_C}{h^2} + \left(1 - \frac{h}{2R}\right) \frac{p_S - p_C}{h^2} + \left(\frac{R_0}{R}\right)^2 \frac{p_E - 2p_C + p_W}{h^2} + \frac{p_T - 2p_C + p_B}{h^2} + k^2 p_C = 0. \quad (23)$$

The conditions in the cylindrical part are found by letting R tend to infinity while $\xi = R/R_0 \rightarrow 1$. At the connection of the two areas where a torus joins a cylinder, the conditions are more complicated. The terms of the Laplacian are obtained from the flow through the boundary planes of a control volume formed around the central point.^{14,15} For example, for

a western torus connected to an eastern cylinder, the first three terms of Eq. (23) change into

$$\begin{aligned}
& \left[1 + \xi \left(1 + \frac{h}{2R} \right) \right] \frac{p_N - p_C}{(1 + \xi)h^2} + \dots \\
& \left[1 + \xi \left(1 - \frac{h}{2R} \right) \right] \frac{p_S - p_C}{(1 + \xi)h^2} + \dots \\
& 2 \frac{p_E - p_C}{(1 + \xi)h^2} + \left(\frac{2}{\xi} \right) \frac{p_W - p_C}{(1 + \xi)h^2}.
\end{aligned} \tag{24}$$

Studied were the situations with ideal open end and closed end. In the first case the pressure is set to zero at the pipe end. In the second case the velocity is zero. To find the condition at the boundary consider point C to be located in the boundary on the right-hand side of the cylinder (Fig. 2). The pressure in a point $2h$ to the west of point C is denoted p_{2W} . For the first and second derivatives in the axial direction p' and p'' we have^{16,17}

$$p' = (p_{2W} - 4p_W + 3p_C)/2h, \tag{25}$$

$$p'' = (p_{2W} - 2p_W + p_C)/h^2. \tag{26}$$

The condition of zero velocity is $p' = 0$. Eliminating p_{2W} from the two equations gives

$$p'' = (2p_W - 2p_C)/h^2. \tag{27}$$

This result can also be obtained by imagining a virtual point E which acts as a mirror of point W in which the pressure is the same. More elaborate expressions are possible;^{16,17} they were tried, but were not found to give better results. Similar expressions are used at the side walls, where the flow is parallel to the boundaries. The mesh dimensions can be adapted to the curved walls, but for simplicity it was preferred to use the same mesh and stepwise to approximate the curved walls. The resulting ragged boundaries cause fluctuations which vary depending on the number of points.

The resulting set of equations are arranged in a matrix and solved on a computer. From the ratio of pressure and velocity at the entrance the input impedance is calculated; mean values are taken over the cross-sections at distances 0 , h and $2h$ from the entrance. The velocity at the input, 0 , is proportional to the local gradient, which is obtained by applying

Eq. (25) to the pressure values at positions 0,1 and 2, giving $p'_0 = (-3p_0 + 4p_1 - p_2)/2h$. The number of points was increased up to the capacity limit of the computer, about 40000 points. The results were plotted against the reciprocal number of points. The irregular fluctuations due to the rectangle-filled circular cross-section were leveled; the error due to the limited number of points was minimized by extrapolation to zero, *i.e.* to an infinite number of points.

The effect of the bend was found by comparing two pipe combinations of the same length, one without and one with a bend. The frequency was varied until either the input pressure or the input velocity was zero, corresponding to zero or infinite impedance. Although the frequency of the straight pipe can also be determined analytically, the FDM results were preferred, since the limited number of grid points resulted in small systematic errors; they were assumed to be the same for straight and curved pipes. The frequency corrections were recalculated in length corrections with Eq. (5) and referred to the tube radius a . At higher frequencies the wavelength is no longer large compared to the grid size, reducing the accuracy. The accuracy of the final result is estimated to be $0.02a$ at the lowest, increasing up to $0.1a$ at the highest frequencies.

III. RESULTS

Consider a duct combination with the same dimensions as the duct drawn in Fig. 1: $l_u/a = 3$, $l_b/a = 4\pi/3$, $l_d/a = 2$, with an ideal open end at the output: $p(s = L, r, \theta) = 0$. Fig. 3 shows the impedance at the input end of the duct as a function of the frequency, and the corresponding length correction, together with the length corrections calculated with Eq. (19) and the FDM. This plot clearly points out the strong dependence with frequency of the mistuning induced by the bending. As reported in earlier studies,²⁻⁶ the length correction at low frequencies is negative. When increasing the frequency, it becomes smaller in absolute value (some oscillations can be observed, they are discussed in the following), to reach zero at a frequency close to $f_c/2$. Then its sign changes and