vidual values may be useful in adjusting, for example, transient response and bandwidth.

Linear algebraic equations also result when an additional zero is added. The equations have the form

$$\begin{aligned} a_{11}(z_1 + z_2) + a_{12}(\pi_1 + \pi_2) &= k_1 \\ a_{21}(z_1 + z_2) + a_{22}(\pi_1 + \pi_2) &= k_2. \end{aligned} \tag{62}$$

V. Conclusions

An algebraic synthesis method based on measurements on the inverse root-locus plot has been presented here. The method differs from others chiefly in that it results in an open-loop transfer function containing prescribed poles. It can further meet specification of velocity constant K_v.

As presented, the method requires that the poles of $G_{op}(s)$ be real. This restriction results from the fact that we can always make a segment of the real-axis part of the inverse root locus. Complex open-loop poles would require not only the condition that gain K be equal to the same value at each, but would also require assurance that the inverse root locus pass through the designated points. More work is needed here to remove the real-pole restriction.

The method, when applied to systems with as many as four poles, yields linear algebraic equations. The linearity will be lost for higher-order systems unless more constraints like the one on K_v are invoked. More work is needed to determine conditions leading to linear algebra for higher-order systems.

Complex-Curve Fitting*

E. C. LEVY†

Summary—**The mathematical analysis of linear dynamic systems, based on experimental test results, often requires that the frequency response of the system be fitted by an algebraic expression. The form in which this expression is usually desired is that of a ratio of two frequency-dependent polynomials.**

In this paper, a method of evaluation of the polynomial coefficients is presented. It is based on the minimization of the weighted sum of the squares of the errors between the absolute magnitudes of the actual function and the polynomial ratio, taken at various values of frequency (the independent variable).

The problem of the evaluation of the unknown coefficients is reduced to that of the numerical solution of certain determinants. The elements of these determinants are functions of the amplitude ratio and phase shift, taken at various values of frequency. This form of solution is particularly adaptable to digital computing methods, bebecause of the simplicity in the required programming. The treatment is restricted to systems which have no poles on the imaginary axis; *i.e.*, to systems having a finite, steady-state (zero frequency) magnitude.

Introduction

In the mathematical treatment of linear dynamic systems, it is usually quite advantageous to deal in the frequency domain rather than the time. In such cases, the behavior or "response" of the system to sinusoidal inputs over a band of frequencies must be known. If the dynamic system under consideration is a simple one, this characteristic of the system, or "transfer function", may be obtained analytically to a reasonable degree of accuracy. If, however, (as in most cases), the system is complex or has a large number of components, it is preferable and more reliable to define the transfer function on the basis of test results. This is done, for example, in the case of an electrical network as shown in Fig. 1, by imposing a sinusoidal voltage of known magnitude and frequency at the input end of the network, and measuring the magnitude and phase of the output voltage. Thus, for the example of Fig. 1, we could define the input and output voltages as:

$$\theta_i = |e_i| \sin(\omega t + \Phi_j) \tag{1}$$

and

$$\theta_0 = |e_0| \sin(\omega t + \Phi_0), \tag{2}$$

respectively. The transfer function of the electrical network shown, defined as the output-per-unit input would be defined by two functions, namely:

a) the amplitude ratio $E_0(\omega)/E_i(\omega)$, and

b) the phase shift $\phi_0(\omega) - \phi_i(\omega) = \Delta\phi(\omega)$, both of which vary with frequency.

In the case of the simple electrical network shown in Fig. 1, the functions $E_0(\omega)/E_i(\omega)$ and $\Delta\phi(\omega)$ can be easily obtained by established analytical methods. However, if the circuit were elaborate, experimental techniques would be found more convenient and reliable. In such case, the functions would be known in graphical form only as shown, for example, in Fig. 2.

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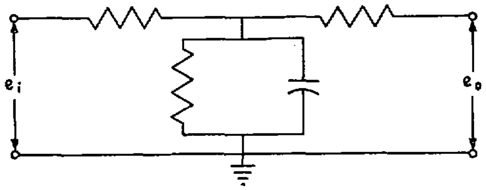

Fig. 1—Simple electrical network showing input and output terminals.

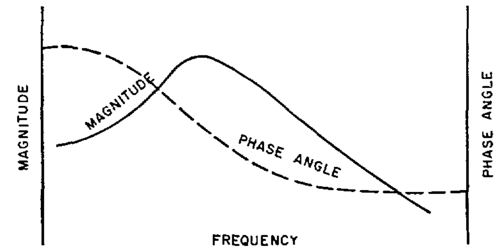

Fig. 2—Frequency response characteristics of a dynamic system.

To further the mathematical analysis of such a system, it becomes desirable to fit the curves of Fig. 2 by an algebraic expression of form suitable for further treatment. The preferred form is that of the ratio of two frequency-dependent polynomials, namely

$$G(j\omega) = \frac{A_0 + A_1(j\omega) + A_2(j\omega)^2 + A_3(j\omega)^3 + \cdots}{B_0 + B_1(j\omega) + B_2(j\omega)^2 + B_3(j\omega)^3 + \cdots}, \quad (3)$$

this form being amenable to linear transform methods of solution. In the following section, a procedure is described which leads to a $G(j\omega)$ of the above form possessing a certain minimum property.

Theory

For convenience in the manipulation of the ensuing work, (3) is rewritten in the following forms:

$$G(j\omega) = \frac{(A_0 - A_2\omega^2 + A_4\omega^4 - \cdots) + j\omega(A_1 - A_3\omega^2 + A_5\omega^4 - \cdots)}{(B_0 - B_2\omega^2 + B_4\omega^4 - \cdots) + j\omega(B_1 - B_3\omega^2 + B_5\omega^4 - \cdots)} \quad (3a)$$

$$= \frac{\alpha + j\omega\beta}{\sigma + j\omega\tau} \quad (3b)$$

$$= \frac{N(\omega)}{D(\omega)} \quad (3c)$$

with the restriction that B_0 be equal to unity.[1]

Suppose now that the function $F(j\omega)$ is used to designate the "ideal" function; *i.e.*, one which represents the data exactly. $F(j\omega)$ will then also have real and imaginary components which would coincide exactly with the values indicated by the experimental curve; *i.e.*,

$$F(j\omega) = R(\omega) + jI(\omega). \quad (4)$$

[1] This "restriction" is merely a matter of convenience. It does not affect the function in any manner; that is, it is not a restriction in the literal sense of the word.

The numerical difference between the two functions $G(j\omega)$ and $F(j\omega)$ represents the error in fitting, that is

$$\epsilon(\omega) = F(j\omega) - G(j\omega) \quad (5a)$$

$$= F(j\omega) - \frac{N(\omega)}{D(\omega)}. \quad (5b)$$

Multiplying both sides of equation (5b) by $D(\omega)$:

$$D(\omega)\epsilon(\omega) = D(\omega)F(j\omega) - N(\omega). \quad (6)$$

The right side of (6) is a function of real and imaginary terms, which may be separated to give:

$$D(\omega)\epsilon(\omega) = a(\omega) + jb(\omega) \quad (7)$$

where $a(\omega)$ and $b(\omega)$ are functions, not only of the frequency, but also of the unknown coefficients A_i and B_i. The magnitude, or absolute value of this function is:

$$|D(\omega)\epsilon(\omega)| = |a(\omega) + jb(\omega)| \quad (8a)$$

$$= \sqrt{a^2(\omega) + b^2(\omega)}. \quad (8b)$$

Then, at any specific value of frequency:

$$|D(\omega_k)\epsilon(\omega_k)|^2 = a^2(\omega_k) + b^2(\omega_k). \quad (9)$$

Let us now define E as being the function given in (9), summed over the sampling frequencies ω_k. Thus:

$$E = \sum_{k=0}^{m} [a^2(\omega_k) + b^2(\omega_k)]. \quad (10)$$

The unknown polynomial coefficients A_i and B_i are now evaluated on the basis of minimizing the function E. [It is this property which characterizes the proposed $G(j\omega)$.] To do so, we first proceed to rewrite (10) in the following form:

$$E = \sum_{k=0}^{m} [(R_k\sigma_k - \omega_k\tau_k I_k - \alpha_k)^2 + (\omega_k\tau_k R_k + \sigma_k I_k - \omega_k\beta_k)^2] \quad (11)$$

making use of (3b) and (4).

Following the accepted standard mathematical procedures, (11) is now differentiated with respect to each of the unknown coefficients A_i and B_i, and the results set equal to zero.

$$
\left.\begin{aligned}
\frac{\partial E}{\partial A_0} &= \sum_{k=0}^{m} - 2(\sigma_k R_k - \omega_k \tau_k I_k - \alpha_k) = 0\\
\frac{\partial E}{\partial A_1} &= \sum_{k=0}^{m} - 2\omega_k(\omega_k \tau_k R_k + \sigma_k I_k - \omega_k \beta_k) = 0\\
\frac{\partial E}{\partial A_2} &= \sum_{k=0}^{m} + 2\omega_k^2(\sigma_k R_k - \omega_k \tau_k I_k - \alpha_k) = 0\\
\frac{\partial E}{\partial A_3} &= \sum_{k=0}^{m} + 2\omega_k^3(\omega_k \tau_k R_k + \sigma_k I_k - \omega_k \beta_k) = 0\\
&\vdots\\
\frac{\partial E}{\partial B_1} &= \sum_{k=0}^{m} - 2\omega_k I_k(\sigma_k R_k - \omega_k \tau_k I_k - \alpha_k) + 2\omega_k R_k(\omega_k \tau_k R_k + \sigma_k I_k - \omega_k \beta_k) = 0\\
\frac{\partial E}{\partial B_2} &= \sum_{k=0}^{m} - 2\omega_k^2 R_k(\sigma_k R_k - \omega_k \tau_k I_k - \alpha_k) - 2\omega_k^2 I_k(\omega_k \tau_k R_k + \sigma_k I_k - \omega_k \beta_k) = 0\\
\frac{\partial E}{3B_3} &= \sum_{k=0}^{m} + 2\omega_k^3 I_k(\sigma_k R_k - \omega_k \tau_k I_k - \alpha_k) - 2\omega_k^3 R_k(\omega_k \tau_k R_k + \sigma_k I_k - \omega_k \beta_k) = 0\\
&\vdots
\end{aligned}\right\} \tag{12}
$$

In the resulting equations, the terms involving the unknown coefficients may be isolated by alluding to the following linear transformations:

$$\alpha_k = A_0 - \alpha_k' \tag{13a}$$

$$\beta_k = A_1 - \beta_k' \tag{13b}$$

$$\left.\begin{aligned} \sigma_k &= B_0 - \sigma_k' = 1 - \sigma', \\ \text{since} \quad B_0 &= 1 \end{aligned}\right\} \tag{13c}$$

$$\tau_k = B_1 - \tau_k'. \tag{13d}$$

Eqs. (12) may thus be rewritten as:

$$
\left.\begin{aligned}
&\sum_{k=0}^{m} + A_0 - \alpha_k' + R_k\sigma_k' + \omega_k I_k B_1 - \omega_k I_k \tau_k' = \sum_{k=0}^{m} R_k\\
&\sum_{k=0}^{m} \omega_k^2(A_1 - \beta_k') + \omega_k I_k \sigma_k' - \omega_k^2 R_k(B_1 - \tau_k') = \sum_{k=0}^{m} \omega_k I_k\\
&\sum_{k=0}^{m} \omega_k^2 R_k \sigma_k' + \omega_k^3 I_k(B_1 - \tau_k') + \omega_k^2(A_0 - \alpha_k') = \sum_{k=0}^{m} \omega_k^2 R_k\\
&\sum_{k=0}^{m} -\omega_k^4 R_k(B_1 - \tau_k') + \omega_k^3 I_k \sigma_k' + \omega_k^4(A_1 - B_k') = \sum_{k=0}^{m} \omega_k^3 I_k\\
&\quad\vdots\\
&\sum_{k=0}^{m} \omega_k I_k(A_0 - \alpha_k') - \omega_k^2 R_k(A_1 - \beta_k') + \omega_k^2(R_k^2 + I_k^2)(B_1 - \tau_k') = 0\\
&\sum_{k=0}^{m} \omega_k^2 R_k(A_0 - \alpha_k') + \omega_k^3 I_k(A_1 - \beta_k') + \omega_k^2(R_k^2 + I_k^2)\sigma_k' = \sum_{k=0}^{m} \omega_k^2(R_k^2 + I_k^2)\\
&\sum_{k=0}^{m} \omega_k^3 I_k(A_0 - \alpha_k') - \omega_k^4 R_k(A_1 - \beta_k') + \omega_k^4(R_k^2 + I_k^2)(B_1 - \tau_k') = 0\\
&\quad\vdots
\end{aligned}\right\}.
$$

Each of (14) will contain terms which are functions of the unknown coefficients, and terms which are known.

To condense the notation before expanding the above equations, the following relationships are defined:

$$\lambda_h = \sum_{k=0}^{m} \omega_k{}^h \tag{15}$$

$$S_h = \sum_{k=0}^{m} \omega_k{}^h R_k \tag{16}$$

$$T_h = \sum_{k=0}^{m} \omega_k{}^h I_k \tag{17}$$

$$U_h = \sum_{k=0}^{m} \omega_k{}^h (R_k{}^2 + I_k{}^2). \tag{18}$$

Substituting these relationships into (14) and separating the coefficients, we obtain the following set of equations:

$$\left.\begin{array}{l}
A_0\lambda_0 - A_2\lambda_2 + A_4\lambda_4 - A_6\lambda_6 + \cdots + B_1T_1 + B_2S_2 - B_3T_3 - B_4S_4 + B_5T_5 + \cdots = S_0 \\
A_1\lambda_2 - A_3\lambda_4 + A_5\lambda_6 - A_7\lambda_8 + \cdots - B_1S_2 + B_2T_3 + B_3S_4 - B_4T_5 - B_5S_6 + \cdots = T_1 \\
A_0\lambda_2 - A_2\lambda_4 + A_4\lambda_6 - A_6\lambda_8 + \cdots + B_1T_3 + B_2S_4 - B_3T_5 - B_4S_6 + B_5T_7 + \cdots = S_2 \\
A_1\lambda_4 - A_3\lambda_6 + A_5\lambda_8 - A_7\lambda_{10} + \cdots - B_1S_4 + B_2T_5 + B_3S_6 - B_4T_7 - B_5S_8 + \cdots = T_3 \\
\vdots \\
A_0T_1 - A_1S_2 - A_2T_3 + A_3S_4 + A_4T_5 - \cdots + B_1U_2 - B_3U_4 + B_5U_6 - B_7U_8 + \cdots = 0 \\
A_0S_2 + A_1T_3 - A_2S_4 - A_3T_5 + A_4S_6 + \cdots + B_2U_4 - B_4U_6 + B_6U_8 - B_8U_{10} + \cdots = U_2 \\
A_0T_3 - A_1S_4 - A_2T_5 + A_3S_6 + A_4T_7 - \cdots + B_1U_4 - B_3U_6 + B_5U_8 - B_7U_{10} + \cdots = 0 \\
\vdots
\end{array}\right\}. \tag{19}$$

Or, in matric notation:

$$(M)\cdot(N) = (C) \tag{20}$$

where

$$(M) = \left\{\begin{array}{llllllllll}
\lambda_0 & 0 & -\lambda_2 & 0 & \lambda_4 \cdots & T_1 & S_2 & -T_3 & -S_4 & T_5 \cdots \\
0 & \lambda_2 & 0 & -\lambda_4 & 0 \cdots & -S_2 & T_3 & S_4 & -T_5 & -S_6 \\
\lambda_2 & 0 & -\lambda_4 & 0 & \lambda_6 \cdots & T_3 & S_4 & -T_5 & -S_6 & T_7 \\
0 & \lambda_4 & 0 & -\lambda_6 & 0 \cdots & -S_4 & T_5 & S_6 & -T_7 & -S_8 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T_1 & -S_2 & -T_3 & S_4 & T_5 \cdots & U_2 & 0 & -U_4 & 0 & U_6 \cdots \\
S_2 & T_3 & -S_4 & -T_5 & S_6 \cdots & 0 & U_4 & 0 & -U_6 & 0 \\
T_3 & -S_4 & -T_5 & S_6 & T_7 \cdots & U_4 & 0 & -U_6 & 0 & U_8 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right\} \tag{21a}$$

and

$$(N) = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ \cdot \\ \cdot \\ B_1 \\ B_2 \\ B_3 \\ \cdot \\ \cdot \end{Bmatrix} \quad (21b) \qquad (C) = \begin{Bmatrix} S_0 \\ T_1 \\ S_2 \\ T_3 \\ \cdot \\ \cdot \\ 0 \\ U_2 \\ 0 \\ \cdot \\ \cdot \end{Bmatrix}. \quad (21c)$$

The numerical value of the unknown coefficients may thus be obtained from (20) once the matrices (21a)–(21c) have been evaluated.

Examples

Example 1) Consider the frequency response function shown in Fig. 3, representing the dynamic characteristics of an arbitrary system. The frequency function from which the curve was drawn is:

$$F(j\omega) = \frac{1 + (j\omega)}{1 + 0.1(j\omega) + 0.01(j\omega)^2}. \quad (22)$$

Table I presents the arbitrary values selected from Fig. 3, to be used as inputs to the program.

The function chosen for the curve-fitting process is:

$$G(j\omega) = \frac{A_0 + A_1(j\omega) + A_2(j\omega)^2}{1 + B_1(j\omega) + B_2(j\omega)^2}. \quad (23)$$

This choice is indicated by the general shape of the curves[2] presented in Fig. 3. The procedure for the numerical evaluation of the unknown coefficients is now as follows:[3] 1) Define the matrices (M), (N), and (C). Thus, for this example, they take the following form:

$$(M) = \begin{Bmatrix} \lambda_0 & 0 & -\lambda_2 & T_1 & S_2 \\ 0 & \lambda_2 & 0 & -S_2 & T_3 \\ \lambda_2 & 0 & -\lambda_4 & T_3 & S_4 \\ T_1 & -S_2 & -T_3 & U_2 & 0 \\ S_2 & T_3 & -S_4 & 0 & U_4 \end{Bmatrix},$$

$$(N) = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix}, \qquad (C) = \begin{Bmatrix} S_0 \\ T_1 \\ S_2 \\ 0 \\ U_2 \end{Bmatrix}.$$

[2] In most cases, the order of the polynomial expression $G(j\omega)$ can be determined from a consideration of the slopes of the magnitude curve, and the phase angle. See J. G. Truxal, "Control System Synthesis," McGraw-Hill Book Co., Inc., New York, N.Y., pp. 350–375; 1955, and G. J. Thaler and R. G. Brown, "Servomechanism Analysis," McGraw-Hill Book Co., Inc., New York, N.Y., pp. 243–249; 1953.

[3] This procedure does not have to be followed for every problem if the equations are programmed for digital computer solution.

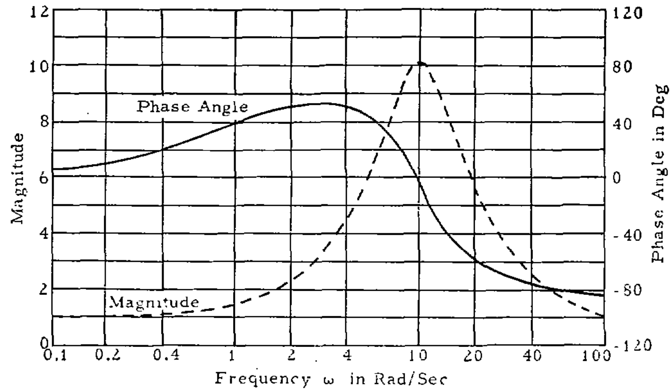

Fig. 3—Frequency response characteristics of a dynamic system with a transfer function given as:

$$F(j\omega) = \frac{1 + j\omega}{1 + 2(0.5)\frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2}.$$

TABLE I

k	ω_k	magnitude	phase angle	R_k	I_k
0	0.0	1.00	0	1.00	0.000
1	0.1	1.00	5	1.00	0.090
2	0.2	1.02	10	1.00	0.177
3	0.5	1.12	24	1.02	0.450
4	0.7	1.24	31	1.05	0.630
5	1.0	1.44	39	1.10	0.900
6	2.0	2.27	51.5	1.41	1.78
7	4.0	4.44	50.5	2.82	3.42
8	7.0	8.17	28	7.23	3.82
9	10.0	10.05	−6	10.00	−1.00
10	20.0	5.56	−59	2.85	−4.77
11	40.0	2.55	−76	0.602	−2.51
12	70.0	1.45	−82	0.188	−1.43
13	100.0	1.00	−84	0.091	−1.01

R_k = (Magnitude at ω_k) × cos (phase angle at ω_k)
I_k = (Magnitude at ω_k) × sin (phase angle at ω_k).

2) Evaluate the λ's, S's, T's, and U's. 3) Substitution in (20) gives five equations with five unknowns, which can be readily solved for each of the unknowns (A's and B's).

For this example, the numerical evaluation of the coefficient from (20) was carried out to eight significant figures, to reduce the effect of computing errors. The results thus obtained are given to five significant figures as follows:

$$A_0 = 0.99936 \qquad B_0 = 1.0000$$
$$A_1 = 1.0086 \qquad B_1 = 0.10097$$
$$A_2 = -0.000015983 \qquad B_2 = 0.010031.$$

By evaluating the function $G(j\omega)$ using these coefficients, it will be observed that the curve of Fig. 3 is fitted well within reading accuracy for the range $0 \leq \omega \leq 100$ rad/second, as required.

The problem of non-minimum phase systems is considered in the following example.

Example 2) Consider the frequency response function illustrated in Fig. 4. It is a graph of the function

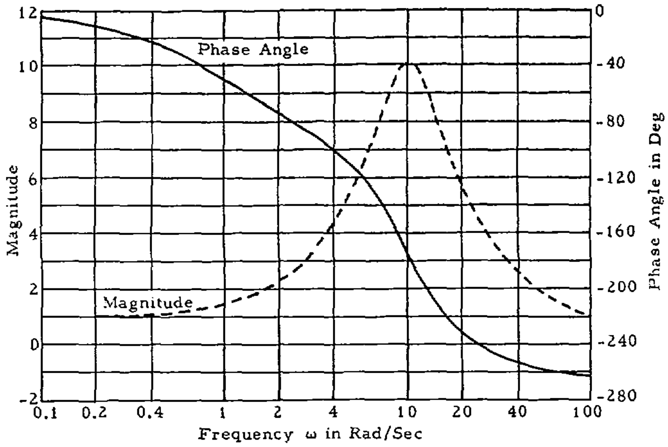

Fig. 4—Frequency response characteristics of a dynamic system with a transfer function given as:

$$F(j\omega) = \frac{1 - j\omega}{1 + 2(0.5)\frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2}.$$

TABLE II

k	ω_k	magnitude	phase angle	R_k	I_k
0	0.0	1.00	0	1.00	0.000
1	0.1	1.00	− 6.5	1.00	−0.113
2	0.2	1.02	− 12.5	1.00	−0.220
3	0.5	1.12	− 29.5	0.975	−0.550
4	0.7	1.24	− 39.0	0.963	−0.780
5	1.0	1.44	− 51.0	0.905	−1.12
6	2.0	2.27	− 75.0	0.588	−2.20
7	4.0	4.44	−102.0	− 0.925	−4.34
8	7.0	8.17	−136.0	− 5.87	−5.69
9	10.0	10.05	−174.0	−10.00	−1.05
10	20.0	5.56	−233.5	− 3.31	4.46
11	40.0	2.55	−253.0	− 0.724	2.44
12	70.0	1.45	−261.0	− 0.227	1.43
13	100.0	1.00	−263.5	− 0.113	0.993

$$F(j\omega) = \frac{1 - j\omega}{1 + 0.1(j\omega) + 0.01(j\omega)^2}. \tag{24}$$

Table II presents the values derived from Fig. 4, and used as inputs to the digital program.

The function chosen for the curve-fitting process was the same one as before, namely that given by (23).

The numerical evaluation of the coefficients was carried out to eight significant figures, as before. The results are presented to five significant figures as follows:

$$A_0 = 0.99741 \qquad B_0 = 1.0000$$
$$A_1 = -0.99483 \qquad B_1 = 0.099607$$
$$A_2 = -0.000020400 \qquad B_2 = 0.0099847.$$

These values also represent the graph of Fig. 4 well within reading accuracy, and for a frequency range well beyond that indicated or required.

Discussion

1) Probably the most essential factor which must be realized in the application of this method is that it imposes a restriction on the types of frequency response functions that can be fitted. This restriction is such that the frequency response function must represent a system which has a finite zero frequency gain; *i.e.*, no poles at the origin. The function may, however, have zero roots; *i.e.*, zeros at the origin. The obvious alternative, if one wishes to apply this method to a function which has an infinite gain at zero frequency, is to modify the function by multiplying it by $(j\omega)^n$, n being large enough to reduce the absolute magnitude of the function at zero frequency to a finite value.

Consider, for example, the transfer function

$$F(j\omega) = \frac{1}{j\omega} \tag{25}$$

representing a pure integrator with unit gain. At $\omega = 0$, the magnitude of the function $F(j\omega)$ is undefined. If this function, or its representative graph, were multiplied by $(j\omega)^1$, a new function $F_M(j\omega)$ would be obtained, whose amplitude ratio is unity, and whose phase shift is also a constant, equal to zero degrees. The inputs to the digital computer would now be as presented in Table III.

TABLE III

k	ω_k	R_k	I_k
0	0	1	0
1	0.1	1	0
2	1	1	0
3	10	1	0
4	100	1	0

If we choose

$$G_M(j\omega) = \frac{A_0 + A_1(j\omega) + A_2(j\omega)^2}{1 + B_1(j\omega) + B_2(j\omega)^2} \tag{26}$$

as before, the results are as follows (see Appendix I):

$$A_0 = 1 \qquad B_0 = 1$$
$$A_1 = 0 \qquad B_1 = 0$$
$$A_2 = 0 \qquad B_2 = 0.$$

To obtain the representation of $F(j\omega)$, we merely divide $G_M(j\omega)$ by the same factor used to convert $F(j\omega)$ to $F_M(j\omega)$. Thus, in this case,

$$G(j\omega) = G_M(j\omega) \times \frac{1}{j\omega} = 1 \times \frac{1}{j\omega} = \frac{1}{j\omega}. \tag{27}$$

2) The method of complex-curve fitting as presented in this paper would correspond to a least-squares fit if $|D(\omega)|$ were a constant. In its indicated form, however, the method may be described as a "weighted least-squares fit," the weighting function being $|D(\omega)|^2$.

Due consideration to (8a) will show that the error $|\epsilon(\omega)|$ generally tends to assume a relative maximum when $|D(\omega)|$ is in the neighborhood of its minima. However, a local minimum in $|D(\omega)|$ corresponds to a local maximum in $|F(j\omega)|$. This, therefore, implies that for a given value of ω, the magnitude of the error is nearly proportional to the magnitude of the function. In general, this "restriction" is not of consequence. If it is, however, the error can easily be reduced by selecting a greater number of sample points in the critical region of the curve.

3) In the process of evaluating the coefficients A_i and B_i, one of them can be assigned an arbitrary numerical value. The author chose to define the coefficient B_0 as unity. This choice, however, is not restrictive, and its selection is left to the discretion of the reader. If a different choice is made, (15) should be appropriately modified.

Appendix I

If a frequency response function such as

$$F(j\omega) = 1$$

is to be analyzed, it will be noted at the outset that the characteristic determinant is equal to zero. This leads to an indefinite solution.

A simple expedient which may be used in this case is to modify the values by some small quantity E, and then consider the limit as E approaches zero.

Thus, in this case, let

$$\begin{aligned} S_0 &= \lambda_0 & U_2 &= \lambda_2 + E_1 \\ S_2 &= \lambda_2 & U_4 &= \lambda_4 + E_2 \\ S_4 &= \lambda_4 & T_1 &= T_3 = 0. \end{aligned}$$

Then:

$$M = \begin{vmatrix} \lambda_0 & 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & \lambda_2 & 0 & -\lambda_2 & 0 \\ \lambda_2 & 0 & -\lambda_4 & 0 & \lambda_4 \\ 0 & -\lambda_2 & 0 & (\lambda_2 + E_1) & 0 \\ \lambda_2 & 0 & -\lambda_4 & 0 & (\lambda_4 + E_2) \end{vmatrix},$$

$$C = \begin{vmatrix} \lambda_0 \\ 0 \\ \lambda_2 \\ 0 \\ (\lambda_2 + E_1) \end{vmatrix}.$$

In terms of its minors and of the characteristic determinant, we therefore obtain:

$$A_0 = \frac{|M_{11}|}{|M|} = 1 \qquad B_0 \equiv 1$$

$$A_1 = \frac{|M_{12}|}{|M|} = 0 \qquad B_1 = \frac{|M_{14}|}{|M|} = 0$$

$$A_2 = \frac{|M_{13}|}{|M|} = \frac{E_1}{E_2} = 0 \qquad B = \frac{|M_{15}|}{|M|} = \frac{E_1}{E_2} = 0.$$

In the above, A_2 and B_2 can be made equal to zero since the magnitudes of E_1 and E_2 are arbitrary and can be assigned such values that $E_1 \ll E_2$, thus making the ratio E_1/E_2 as close to zero as need be.