

A class of Bézier-like curves

Qinyu Chen*, Guozhao Wang

Department of Mathematics, Zhejiang University, Hangzhou, 310027 People's Republic of China

Received 22 December 2002; received in revised form 22 December 2002; accepted 23 December 2002

Abstract

In this paper, a new basis, to be called C-Bézier basis, is constructed for the space $\Gamma_n = \text{span}\{1, t, t^2, \dots, t^{n-2}, \sin t, \cos t\}$ by an integral approach. Based on this basis, we define C-Bézier curves. We then show that such basis and curves share the same properties as the Bernstein basis and the Bézier curves in polynomial spaces respectively. © 2003 Elsevier Science B.V. All rights reserved.

Keywords: C-Bézier basis; C-Bézier curve; Bézier basis; Bézier curve; Trigonometric polynomials

1. Introduction

The rational Bézier model is a powerful tool for constructing free-form curves and surfaces. It is well known that the Bézier basis is a basis for the space of degree- n algebraic polynomials $T = \text{span}\{1, t, t^2, \dots, t^n\}$. However, it has many shortcomings due to its rational form and polynomial form. For instance, repeated differentiation produces curves of very high degree. In particular, it cannot represent exactly transcendental curves such as the helix and the cycloid. In order to avoid the inconveniences of the rational Bézier model, finding new bases in new spaces seems to be the only way.

Many bases are presented in new spaces other than the polynomial space. Peña (1997) constructed a basis for $C_m = \text{span}\{1, \cos t, \dots, \cos mt\}$. Zhang (1996, 1997) investigated curves in the space $\text{span}\{1, t, \cos t, \sin t\}$. Sánchez-Reyes (1998) gave a basis for the space of trigonometric polynomials $\{1, \sin t, \cos t, \dots, \sin mt, \cos mt\}$. Mainar et al. (2001) found some bases for the spaces $\{1, t, \cos t, \sin t, \cos 2t, \sin 2t\}$, $\{1, t, t^2, \cos t, \sin t\}$, and $\{1, t, \cos t, \sin t, t \cos t, t \sin t\}$. But all these bases do not encompass free-form polynomial curves of high order.

* Corresponding author.
E-mail address: xiaocin@zju.edu.cn (Q. Chen).

In this paper, we present a new basis, called the C-Bézier basis, by using an integral approach, for the space $\Gamma_n = \text{span}\{1, t, t^2, \dots, t^{n-2}, \cos t, \sin t\}$, in which t^{n-1} and t^n in T are replaced by $\cos t$ and $\sin t$. With this basis, circular arc and polynomial curves of high order can be represented exactly.

The rest of this paper is organized as follows. Section 2 gives an algorithm for constructing the basis. Some properties of the C-Bézier basis are discussed in Section 3. Using this basis, we give the definition of C-Bézier curve. We shall show that such curves display most of the properties of the Bézier curves in Section 4. As an example, we shall give the representation of a circle integrally.

2. Construction of the C-Bézier basis

We first give two initial functions

$$\begin{aligned} u_{0,1}(t) &= \sin(\alpha - t) / \sin \alpha, \\ u_{1,1}(t) &= \sin t / \sin \alpha, \end{aligned} \quad (2.1)$$

where $t \in [0, \alpha]$ and $\alpha \in [0, \pi]$.

Let

$$\delta_{0,1} = \left(\int_0^\alpha u_{0,1}(t) dt \right)^{-1} \quad \text{and} \quad \delta_{1,1} = \left(\int_0^\alpha u_{1,1}(t) dt \right)^{-1}.$$

Define

$$u_{0,2}(t) = 1 - \int_0^t \delta_{0,1} u_{0,1}(s) ds = (1 - \cos(\alpha - t)) / (1 - \cos \alpha),$$

$$u_{1,2}(t) = \int_0^t (\delta_{0,1} u_{0,1}(s) - \delta_{1,1} u_{1,1}(s)) ds = (1 - \cos t + \cos \alpha - \cos(\alpha - t)) / (1 - \cos \alpha),$$

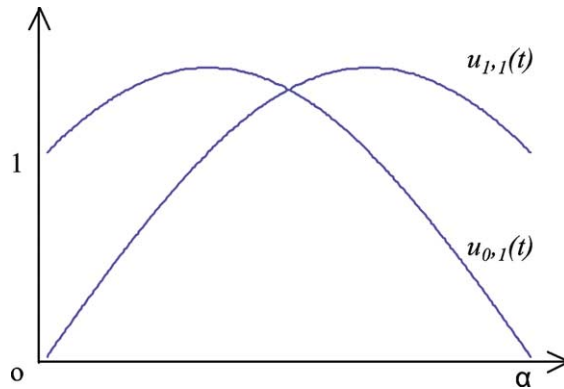


Fig. 1. The two initial functions.

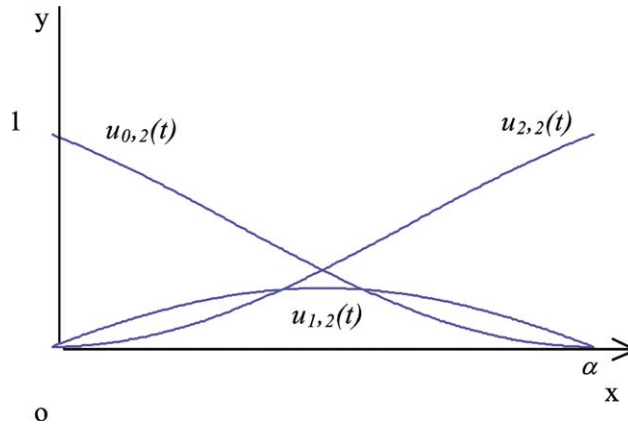


Fig. 2. The quadratic C-Bézier basis.

$$u_{2,2}(t) = \int_0^t \delta_{1,1} u_{1,1}(s) ds = (1 - \cos t) / (1 - \cos \alpha).$$

Definition 2.1. $\{u_{0,2}, u_{1,2}, u_{2,2}\}$ is called the C-Bézier basis for the space $\Gamma_2 = \text{span}\{1, \sin t, \cos t\}$.

In the same way, for $n > 2$ we define the C-Bézier basis $\{u_{0,n}, u_{1,n}, \dots, u_{n,n}\}$ of the space $\Gamma_n = \text{span}\{1, t, \dots, t^{n-2}, \sin t, \cos t\}$ recursively by

$$\begin{aligned} u_{0,n}(t) &= 1 - \int_0^t \delta_{0,n-1} u_{0,n-1}(s) ds, \\ u_{i,n}(t) &= \int_0^t (\delta_{i-1,n-1} u_{i-1,n-1}(s) - \delta_{i,n-1} u_{i,n-1}(s)) ds, \\ u_{n,n}(t) &= \int_0^t \delta_{n-1,n-1} u_{n-1,n-1}(s) ds. \end{aligned} \quad (2.2)$$

In these formulae,

$$\delta_{i,n} = \left(\int_0^\alpha u_{i,n}(t) dt \right)^{-1}, \quad i = 0, 1, \dots, n.$$

In particular, when $n = 3$, this is just the basis presented by Zhang (1996, 1997). The basis of Mainar et al. (2001) presented in the space $\text{span}\{1, t, t^2, \cos t, \sin t\}$ is exactly $\{u_{0,4}, u_{1,4}, u_{2,4}, u_{3,4}, u_{4,4}\}$.

In fact, if we use $u_{0,1}(t) = 1 - t$ and $u_{1,1}(t) = t$ as the two initial functions, we get the Bézier basis for the polynomial space from (2.2).

3. Properties of the basis

3.1. Properties at the endpoints

Lemma 3.1. *At the endpoints, the C-Bézier basis has the same properties as the Bézier basis. That is, for $n \geq 2$,*

$$(1) \quad u_{0,n}(0) = u_{n,n}(\alpha) = 1, \quad (3.1)$$

$$(2) \quad u_{i,n}^{(j)}(0) = u_{i,n}^{(k)}(\alpha) = 0, \quad j = 0, 1, \dots, i-1, \quad k = 0, 1, \dots, n-i-1, \quad (3.2)$$

$$(3) \quad u_{i,n}^{(i)}(0) = \delta_{i-1,n-1} \delta_{i-2,n-2} \cdots \delta_{0,n-i}, \quad i = 1, 2, \dots, n. \quad (3.3)$$

By Definition 2.1 and from (2.2), it is easy to prove Lemma 3.1 by induction on n .

3.2. Linear independence

In order to check the linear independence of $\{u_{0,n}, u_{1,n}, \dots, u_{n,n}\}$, we consider a trivial linear combination $\sum_{i=0}^n \alpha_i u_{i,n}(t) = 0$, $t \in [0, \alpha]$. By taking $t = 0$, we get from (3.2) that $\alpha_0 = 0$. Differentiating the linear combination k times we deduce again from (3.2) that $\alpha_k = 0$ for $k = 1, 2, \dots, n$. That is, $u_{i,n}(t) (i = 0, \dots, n)$ are linear independent. Therefore $(u_{0,n}, u_{1,n}, \dots, u_{n,n})$ is a basis for Γ_n .

3.3. Degree-elevation

Suppose that

$$u_{i,n}(t) = \sum_{j=0}^{n+1} \alpha_{ij} u_{j,n+1}(t). \quad (3.4)$$

Differentiating (3.4) k times, we have from Lemma 3.1 that $\alpha_{ik} = 0$ for $k = 0, 1, \dots, i-1, i+2, \dots, n+1$. Thus

$$u_{i,n}(t) = \alpha_{ii} u_{i,n+1}(t) + \alpha_{i,i+1} u_{i+1,n+1}(t). \quad (3.5)$$

Using L'Hospital's Rule, we have

$$\alpha_{ii} = \frac{u_{i,n}(t) - \alpha_{i,i+1} u_{i+1,n+1}(t)}{u_{i,n+1}(t)} = \lim_{t \rightarrow 0^+} \frac{u_{i,n}(t) - \alpha_{i,i+1} u_{i+1,n+1}(t)}{u_{i,n+1}(t)} = \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)}. \quad (3.6)$$

Similarly, we have

$$\alpha_{i,i+1} = \frac{u_{i,n}^{(n-i)}(\alpha)}{u_{i+1,n+1}^{(n-i)}(\alpha)}. \quad (3.7)$$

From Lemma 3.1, we have

$$\sum_{i=0}^n u_{i,n}(t) = \sum_{i=0}^n (\alpha_{ii} u_{i,n+1}(t) + \alpha_{i,i+1} u_{i+1,n+1}(t)) = 1 = \sum_{i=0}^{n+1} u_{i,n+1}(t).$$

From the linear independence of the C-Bézier basis, we have

$$\alpha_{i,i+1} = \begin{cases} 1 - \alpha_{i+1,i+1}, & i = 0, 1, \dots, n-1, \\ 1, & i = n+1. \end{cases}$$

Thus, we have the degree-elevation formula

$$u_{i,n}(t) = \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)} u_{i,n+1}(t) + \left(1 - \frac{u_{i+1,n}^{(i+1)}(0)}{u_{i+1,n+1}^{(i+1)}(0)}\right) u_{i+1,n+1}(t). \quad (3.8)$$

This is a small generalization of degree-elevation of Bézier basis, in which a basic function is multiplied by $(1-t+t)$.

3.4. Positivity

The positivity of the Bézier basis $\{b_{i,n}(t) = C_n^i t^i (1-t)^{n-i}, t \in [0, 1]\}$ is obvious. This is not apparent, however, for the C-Bézier basis. Zhang (1996) gave the proof for the C-Bézier basis for the space $\Gamma_3 = \text{span}\{1, t, \sin t, \cos t\}$. However, the proof is quite hideous and its method cannot be used for the space Γ_n when $n > 3$. In the following, we give a simple proof.

Lemma 3.2. For $n \geq 2$, let $f = g(\sin t, \cos t, 1, t, \dots, t^{n-2})$ where g is a linear function and t is in $[\alpha, \beta]$. If $\beta - \alpha \leq \pi$, then f has at most n zeros on $[\alpha, \beta]$.

Proof. Suppose that f has $n+1$ zeros on $[\alpha, \beta]$. By the Rolle's Theorem, we have that $f^{(n-1)}$ has two zeros on (α, β) . Since $f = a \sin t + b \cos t + \sum_{i=0}^{n-2} c_i t^i$, there are real numbers a, b, A, B, C, φ such that $f^{(n-1)} = A \sin t + B \cos t = C \sin(t + \varphi)$. However, $\sin(t + \varphi)$ has at most one zero on the interval (α, β) . This contradiction proves that the assumption is false. So f has at most n zeros. \square

Lemma 3.3. The C-Bézier basis are positive on $[0, \alpha]$, $0 \leq \alpha \leq \pi$.

Proof. Consider an arbitrary C-Bézier basic function $u_{i,n}(t)$, $n \geq 2$, $0 \leq i \leq n$. We see in (3.2) that $u_{i,n}(t)$ has n zeros on $[0, \alpha]$, including the i -fold zero at 0 and the $(n-i)$ -fold zero at α . From Lemma 3.2, we conclude that $u_{i,n}(t)$ has no zero on (α, β) . In other words, $u_{i,n}(t)$ is either positive or negative on the interval. We conclude from (3.3) that $u_{i,n}(t)$ is positive on $(0, \alpha)$. Since $u_{i,n}(t)$ is arbitrary, we know that the C-Bézier basis is positive on $[0, \alpha]$, $0 \leq \alpha \leq \pi$. \square

Lemma 3.4. The C-Bézier basis is normalized, that is,

$$\sum_{i=0}^n u_{i,n}(t) = 1.$$

We summarize Lemmas 3.3 and 3.4 in

Proposition 3.1. The C-Bézier basis is a blending system.

3.5. Symmetry

Proposition 3.2. $u_{i,n}(t) = u_{n-i,n}(\alpha - t)$ for $t \in [0, \alpha]$, $i = 0, 1, \dots, n$.

Proof. We prove the equality in this proposition by induction. When $n = 1$, the proposition obviously holds by the definition of the C-Bézier basis. Assume that the property holds for $n = k$, that is,

$$u_{i,k}(t) = u_{k-i,k}(\alpha - t).$$

Hence we have,

$$\int_0^{\alpha-t} u_{k-i,k}(s) ds = - \int_{\alpha}^t u_{k-i,k}(\alpha - s) ds = \int_t^{\alpha} u_{i,k}(s) ds = \delta_{i,k}^{-1} - \int_0^t u_{i,k}(s) ds.$$

By letting $t = 0$, we obtain $\delta_{k-i,k} = \delta_{i,k}$. Therefore, for $1 < i < k + 1$, we have

$$\begin{aligned} u_{k+1-i,k+1}(\alpha - t) &= \delta_{k-i,k} \int_0^{\alpha-t} u_{k-i,k}(s) ds - \delta_{k+1-i,k} \int_0^{\alpha-t} u_{k+1-i,k}(s) ds \\ &= \left(1 - \delta_{i,k} \int_0^t u_{i,k}(s) ds \right) - \left(1 - \delta_{i-1,k} \int_0^t u_{i-1,k}(s) ds \right) \\ &= u_{i,k+1}(t). \end{aligned}$$

The proof for the case when $i = 1$ and $i = k + 1$ is similar. So, the proposition holds by induction on n . \square

4. Geometric properties of the C-Bézier curve

A C-Bézier curve $p(t)$ with control points p_i is defined by

$$p(t) = \sum_{i=0}^n p_i u_{i,n}(t), \quad t \in [0, \alpha], \quad (4.1)$$

where $\{u_{i,n}(t)\}$ is the C-Bézier basis for the space $\Gamma_n = \text{span}\{1, t, \dots, t^{n-2}, \cos t, \sin t\}$, and α is a global shape parameter.

4.1. Geometric properties at the endpoints

The geometric properties at the endpoints of the C-Bézier curves are obvious from those of the C-Bézier basis.

$$(1) \quad p(0) = p_0, \quad p(\alpha) = p_n, \quad (4.2)$$

$$(2) \quad p^{(k)}(0) = \sum_{i=0}^k p_i u_{i,n}^{(k)}(0). \quad (4.3)$$

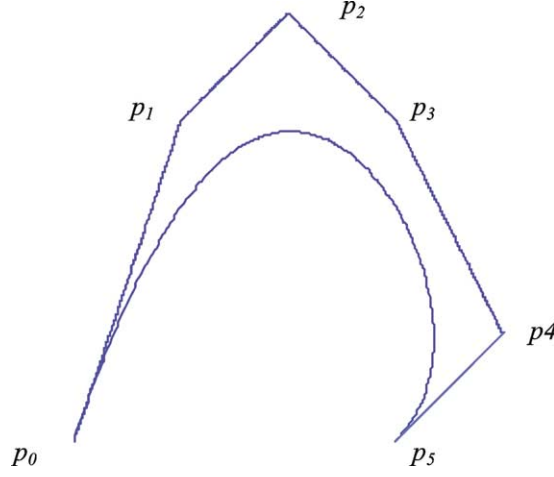


Fig. 3. Convex hull property.

4.2. Convex hull property

The entire C-Bézier curve (4.1) must lie inside its control polygon spanned by p_0, p_1, \dots, p_n . This property is a consequence of Proposition 3.1.

4.3. Degree-elevation

The degree-elevation problem is stated as follows: Given a C-curve of degree n and shape parameter α , we want to express it as a C-curve of degree $n + 1$, that is, find $n + 1$ points $p_i^*, i = 0, 1, \dots, n + 1$, which satisfy

$$p(t) = \sum_{i=0}^n p_i u_{i,n}(t) = \sum_{i=0}^{n+1} p_i^* u_{i,n+1}(t). \quad (4.4)$$

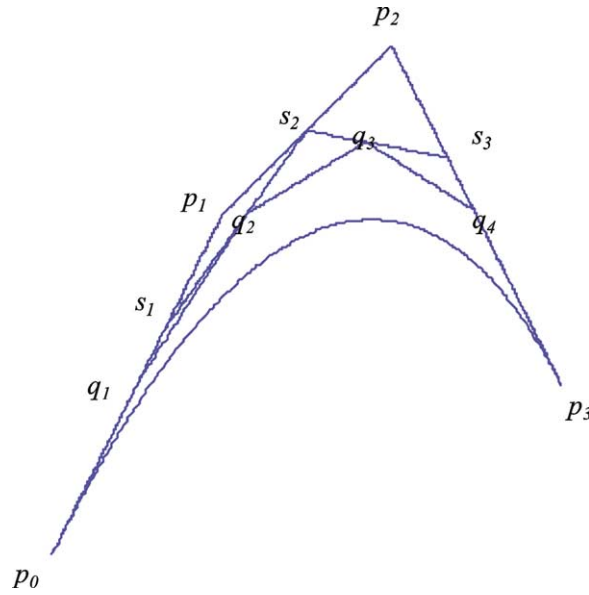
For a curve of the form (4.1), from (3.8) and (4.4), we have

$$\begin{aligned} p(t) &= \sum_{i=0}^n p_i u_{i,n}(t) = \sum_{i=0}^n p_i \left(\frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)} u_{i,n+1}(t) + \left(1 - \frac{u_{i+1,n}^{(i+1)}(0)}{u_{i+1,n+1}^{(i+1)}(0)} \right) u_{i+1,n+1}(t) \right) \\ &= \sum_{i=0}^{n+1} p_i^* u_{i,n+1}(t). \end{aligned}$$

Hence the $n + 1$ points

$$\begin{aligned} p_0^* &= p_0, \\ p_i^* &= \left(1 - \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)} \right) p_{i-1} + \frac{u_{i,n}^{(i)}(0)}{u_{i,n+1}^{(i)}(0)} p_i, \quad i = 1, 2, \dots, n, \\ p_{n+1}^* &= p_n \end{aligned} \quad (4.5)$$

is a degree-elevation of the given curve.

Fig. 4. Degree-elevation ($\alpha = 0.75\pi$).

In fact, a degree-elevation procedure is a corner cutting procedure. The formula (4.5) also holds for the degree-elevation of Bézier curve.

Interested readers can verify that the sequence of control polygons we get recursively by (4.5) converges to the C-Bézier curve.

4.4. The limit of C-Bézier curves

Proposition 4.1. *As $\alpha \rightarrow 0$, the limit of a C-Bézier curve in the space $\Gamma_n = \text{span}\{1, t, t^2, \dots, t^{n-2}, \sin t, \cos t\}$ approaches a Bézier curve in the space $T_n = \text{span}\{1, t, t^2, \dots, t^n\}$.*

Zhang (1996) proved this proposition in the space Γ_3 . Suppose now that it holds in the space Γ_{n-1} . After reparametrizing by $\tau = t/\alpha$, by the inductive hypothesis,

$$u_{i,n-1}(t) = u_{i,n-1}(\alpha\tau) = v_{i,n-1}(\tau), \quad \lim_{\alpha \rightarrow 0} v_{i,n-1}(\tau) = b_{i,n-1}(\tau). \quad (4.6)$$

By (2.2), we have

$$u_{i,n}(t) = \int_0^t (\delta_{i-1,n-1} u_{i-1,n-1}(s) - \delta_{i,n-1} u_{i,n-1}(s)) ds = \frac{\int_0^t u_{i-1,n-1}(s) ds}{\int_0^\alpha u_{i-1,n-1}(s) ds} - \frac{\int_0^t u_{i,n-1}(s) ds}{\int_0^\alpha u_{i,n-1}(s) ds}. \quad (4.7)$$

Note that

$$\int_0^1 b_{i,n}(\tau) d\tau = \frac{1}{n+1}.$$

Hence

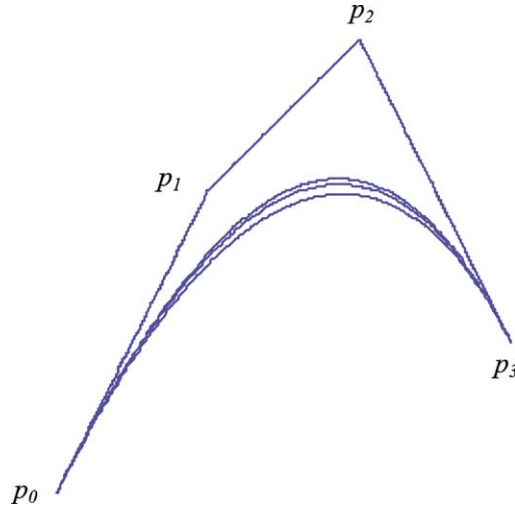


Fig. 5. The limit of the C-Bézier curve.

$$\begin{aligned}
 \lim_{\alpha \rightarrow 0} v'_{i,n}(\tau) &= \lim_{\alpha \rightarrow 0} \left(\frac{v_{i-1,n-1}(\tau)}{\int_0^1 u_{i-1,n-1}(\alpha\tau) d\tau} - \frac{v_{i,n-1}(\tau)}{\int_0^1 u_{i,n-1}(\alpha\tau) d\tau} \right) \\
 &= \frac{b_{i-1,n-1}(\tau)}{\int_0^1 b_{i-1,n-1}(\tau) d\tau} - \frac{b_{i,n-1}(\tau)}{\int_0^1 b_{i,n-1}(\tau) d\tau} \\
 &= n(b_{i-1,n-1}(\tau) - b_{i,n-1}(\tau)) \\
 &= b'_{i,n}(\tau).
 \end{aligned}$$

By (3.1) and (3.2), $u_{i,n}(0) = b_{i,n}(0)$. Therefore

$$\lim_{\alpha \rightarrow 0} v_{i,n}(\tau) = b_{i,n}(\tau). \quad (4.8)$$

From the definitions of C-Bézier curve and Bézier curve, we have Proposition 4.1.

4.5. Differentiation

The derivative $p'(t)$ of a degree- n C-Bézier $p(t)$ is clearly a degree $n-1$ curve. Such a curve can be written in C-Bézier like form as

$$p'(t) = \sum_{i=0}^{n-1} p_i u_{i,n-1}(t), \quad t \in [0, \alpha],$$

where $p_i, i = 0, 1, \dots, n-1$, are the control points of $p'(t)$. Differentiating the functions in (3.3) and after some algebraic manipulations, we find that the control points of $p'(t)$ in the above form are given by

$$\delta_{i,n-1}(p_{i+1} - p_i). \quad (4.9)$$

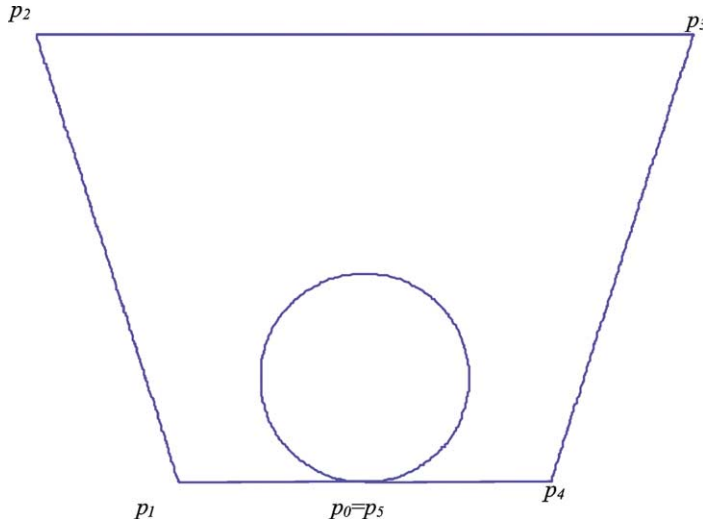


Fig. 6. Representation of a circle by C-Bézier curve.

4.6. Representation of circle integrally

Zhang (1996) gave the representation of a given unitary circular arc spanning an angle α with the C-Bézier curve for Γ_3 . However, the parameter α is restricted to $[0, \pi]$, that is, the C-Bézier curve presented by Zhang can just express a circle spanning the angle α , $\alpha \in [0, \pi]$. However, we can check that the C-Bézier basis for Γ_5 is positive on $[0, \alpha)$, $\alpha \in [0, 2\pi]$. Therefore, the range of parameter α can be extended to $[0, 2\pi]$.

Proposition 4.2. Let $p_0, p_1, p_2, p_3, p_4, p_5$ be six control points which satisfy

- (1) $p_1 p_2 p_3 p_4$ is an isosceles trapezoid,
- (2) $p_0 (= p_5)$ is the midpoint of $p_1 p_4$,
- (3) the height of the trapezoid is $\pi(\pi - 3)/3$,
- (4) $|p_2 p_3| = 2\pi$, $|p_1 p_4| = -2/\pi + 4\pi/3$.

Then the C-Bézier curve $p(t) = \sum_{i=0}^5 p_i u_i$ is a circle where $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ is the C-Bézier basis for the space Γ_5 .

5. C-Bézier surface

Using tensor product, we can construct C-Bézier surface

$$p(s, t) = \sum_{i=0}^m \sum_{j=0}^n b_{i,j} u_{i,m}(s) u_{j,n}(t), \quad 0 \leq \alpha \leq \pi, \quad 0 \leq s, t \leq \alpha,$$

in which $u_{i,m}(s)$, $u_{j,n}(t)$ are the C-Bézier basic functions and b_{ij} is the control point. Tensor product of C-Bézier curves has properties similar to those of tensor product of Bézier curves.

Acknowledgements

We are very grateful to the referees for their helpful suggestions and comments. Also we wish to thank Dr. Zhaozhen Huang for his help in improving the language. This work was partial supported by the National Natural Science Foundation of China (Grant No. 60073023) and Foundation of State Key Basic Research 973 Development Programming Item of China (No. G1998030600).

References

- Mainar, E., Peña, J.M., Sánchez-Reyes, J., 2001. Shape preserving alternatives to the rational Bézier model. *Computer Aided Geometric Design* 18, 37–60.
- Peña, J.M., 1997. Shape preserving representations for trigonometric polynomial curves. *Computer Aided Geometric Design* 14, 5–11.
- Sánchez-Reyes, J., 1998. Harmonic rational Bézier curves, p-Bézier curves and trigonometric polynomials. *Computer Aided Geometric Design* 15, 909–923.
- Zhang, J.W., 1996. C-curves: An extension of cubic curves. *Computer Aided Geometric Design* 13, 199–217.
- Zhang, J.W., 1997. Two different forms of C-B-splines. *Computer Aided Geometric Design* 14, 31–41.