

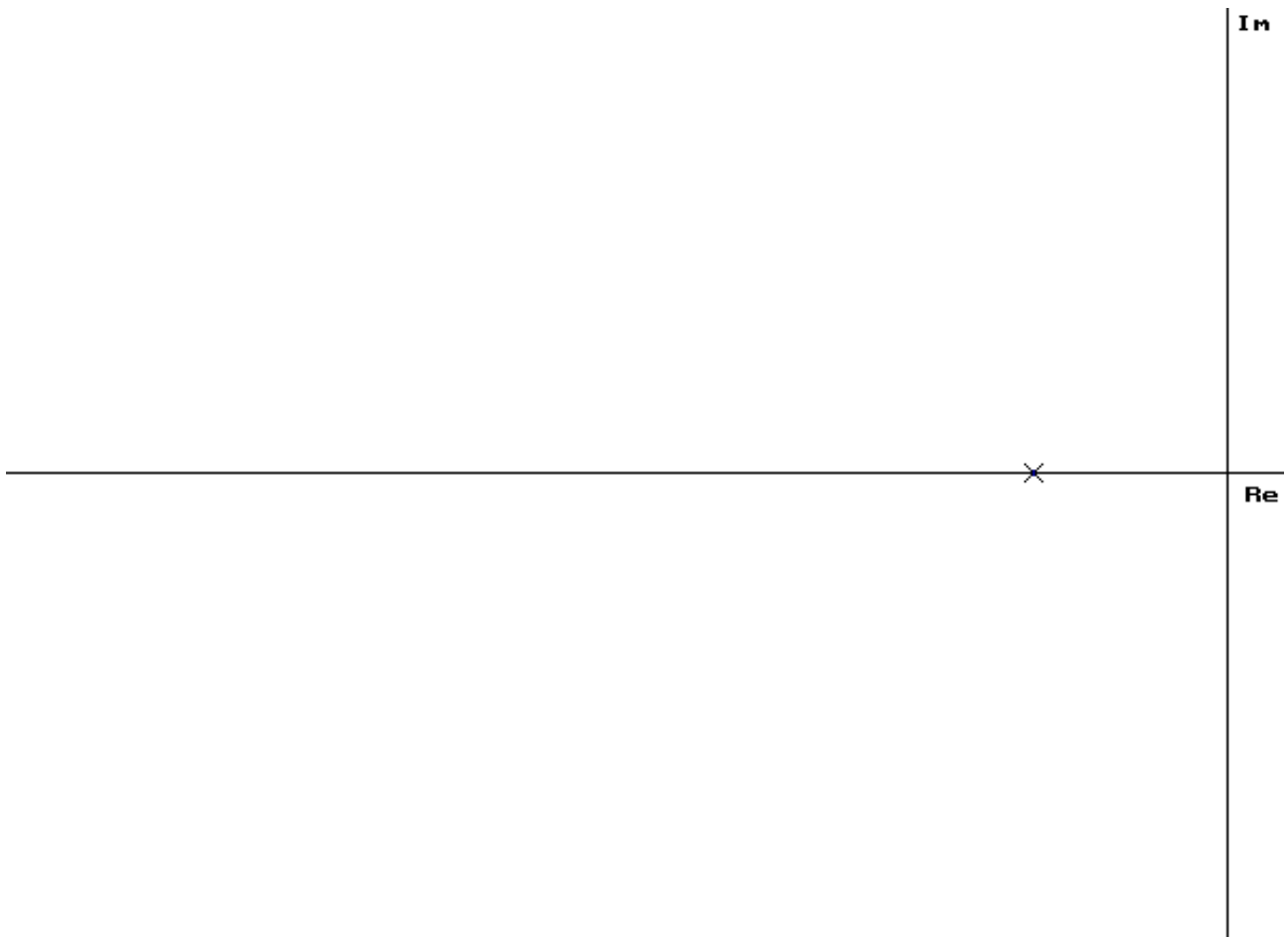
# Relation between all-pass filters and low-pass filters

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Mathematically, a linear time-invariant continuous-time lumped filter can be characterized with a gain factor and two finite sets of complex numbers known as the poles and the zeros. Within the signal range for which they are intended, normal analogue filters are close enough to being linear, time-invariant, continuous-time and lumped to be treated the same way.

As I promised to avoid complex numbers as much as possible, I will only mention that each pole and each zero defines a spot on a two-dimensional plane known as the  $s$  plane. The vertical axis (also known as imaginary axis) of the  $s$  plane corresponds to the radian frequency, that is,  $2\pi$  times the normal frequency. The magnitude response of the filter is proportional to the product of the distances to the zeros divided by the product of the distances to the poles.

This is illustrated in figure 1 for a simple first-order low-pass filter with a pole at  $-1000 \pi$  rad/s and no zeros<sup>1</sup>. When you want to know how the response drops off with frequency, draw a point at the vertical axis at  $2\pi$  times the frequency you are interested in, and calculate the distance to the cross that represents the pole<sup>2</sup>. The magnitude is inversely proportional to that distance.



*Figure 1: Pole-zero pattern of the very simplest low-pass filter: a first-order low-pass with one pole and no zeros. As an example, the pole is supposed to be at  $-1000 \pi$  rad/s.*

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- 1 People who prefer extended complex numbers over ordinary complex numbers would say no finite zeros rather than no zeros, but that's a mathematical detail.
  - 2 Poles are usually drawn as crosses and zeros as circles.

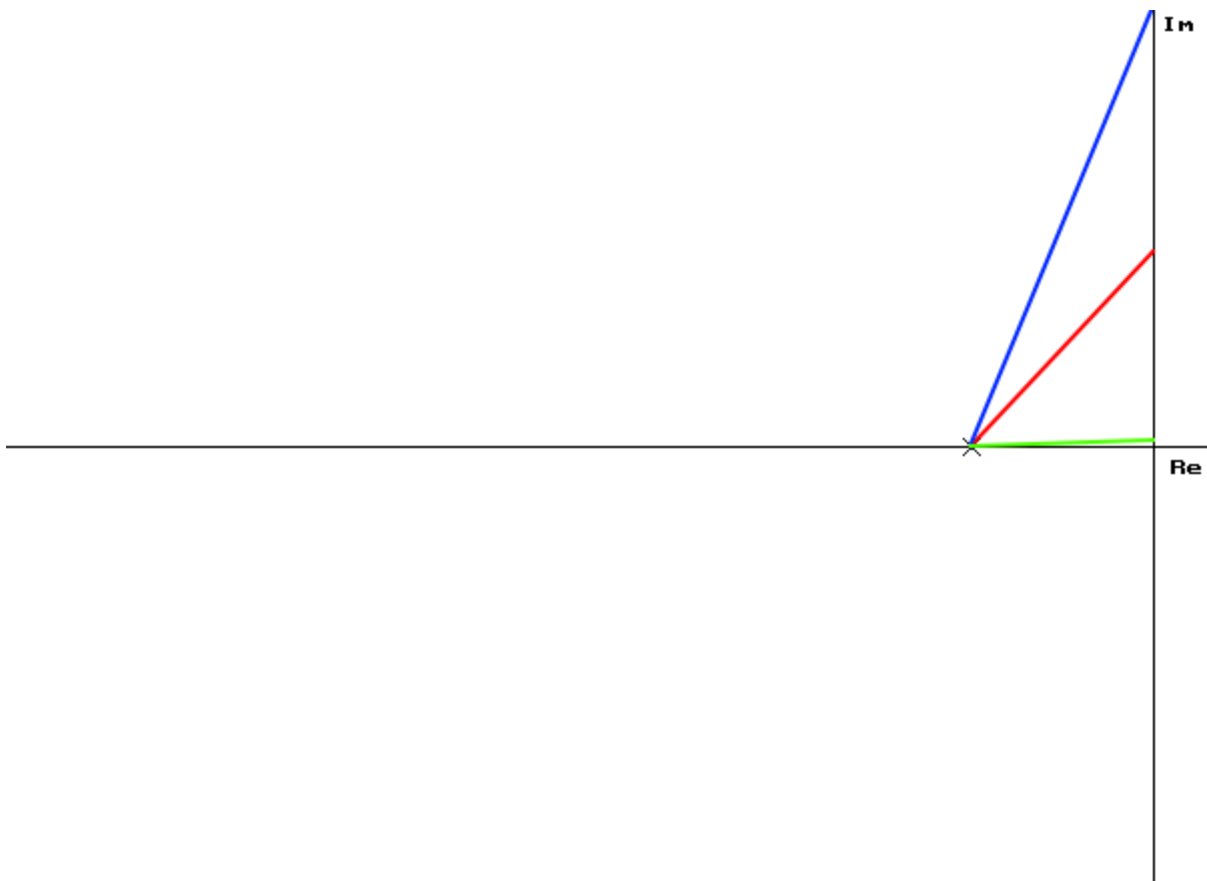


Figure 2: Line segment to the pole at a low frequency (green), a frequency around the -3 dB point (red) and at a high frequency (blue)

Using Pythagoras' theorem, it is clear that the distance increases and the response drops with increasing frequency, as you would expect for a low-pass, and that the distance is precisely  $\sqrt{2}$  times as large at 500 Hz (corresponding to  $1000 \pi$  rad/s) as at 0 Hz. At 500 Hz, the response has therefore dropped to  $1/\sqrt{2}$  times the value for very low frequencies - expressed in decibels, -3.010299... dB, which almost everyone rounds to -3 dB.

The angle of the line between the pole and the spot on the vertical axis relates to the phase shift of the filter. At 0 Hz, the line is horizontal and the phase shift is  $0^\circ$ . At very high frequencies, the line is almost vertical and the phase shift almost  $-90^\circ$ . At 500 Hz, the line is diagonal and the phase shift  $-45^\circ$ .

Compared to poles in the left half plane, zeros in the left half plane have the opposite effect on the phase shift and the reciprocal effect on the magnitude (the further the spot on the vertical axis is from a zero, the bigger the magnitude response).

Zeros in the right half plane have the same effect on the magnitude response as zeros in the left half plane, but the opposite effect on the phase.

Poles in the right half plane have a rather disastrous effect: they make the filter unstable and turn it into either an oscillator or a Schmitt trigger.

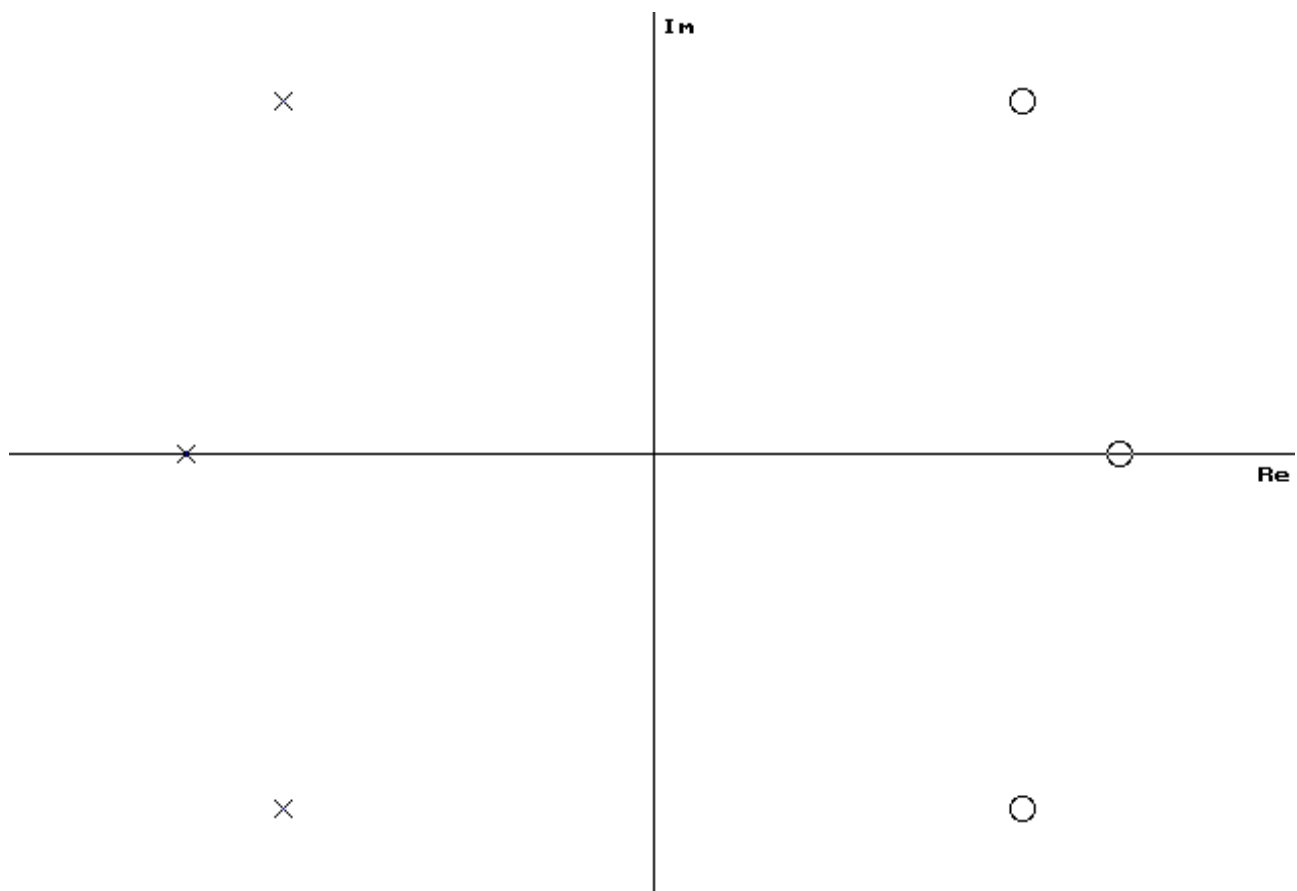
Hence, a pole in the left half plane and a zero in the right half plane, mirrored in the vertical axis, together result in a flat magnitude response and twice the phase shift of the pole.

Bessel low-pass filters are known for their good phase response. Although they are never perfectly linear phase, they can approximate it pretty well up to a certain frequency - see the many tables and graphs about Bessel low-pass filters for details. Filter alignments such as the 0.05° equiripple linear phase filter are similar to Bessel, but somewhat worse at low and better at higher frequencies. These are all filters with only poles and no (finite) zeros.

Hence, you can design an all-pass filter with a pretty flat delay like this:

- Choose a Bessel filter (or a 0.05° equiripple linear phase filter) with 1/2 of the delay that you want to have, up to as high a frequency as you need.
- Convert it into an all-pass by adding the zeros in the right half plane that are the mirror images of the poles.

As an example, figure 3 shows the pole-zero pattern for an all-pass filter based on a third-order Bessel filter.



*Figure 3 Pole-zero pattern of a third-order all-pass filter based on a third-order Bessel low-pass filter. The phase shift and hence the delay is twice that of the low-pass filter.*

Practically, for a filter made of a cascade of second-order sections with or without a first-order section, this means that the all-pass sections must have the same  $Q$ 's and  $\omega_0$ 's as the second-order low-pass sections of the Bessel low-pass would have had. The first-order section must have the same time constant as the first-order section of the Bessel filter would have had.