THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 26, NUMBER 6 NOVEMBER, 1954

Acoustical Studies of the Tractrix Horn. I*†

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(Received June 14, 1954)

When predicting and comparing the acoustical properties of horns it is customary practice to formulate the propagation as a one-parameter plane wave front problem. However, when particular attention is paid to the rapid flare near the mouth of a horn structure such as the tractrix, it also seems plausible to formulate the propagation on the basis of a one-parameter spherical wave front theory. By visualizing the surfaces of constant phase as spheres of constant radii a and the flow lines as tractrixes having a generating arm of length a, a one-parameter wave equation and Ricatti impedance equation may be derived. Solutions to these equations have been obtained by wave perturbation and by analog computer techniques.

Axial response and throat impedance measurements are compared with theoretical calculations postulating first a hemispherical and then a plane piston radiation pattern. It appears that the most satisfactory explanation lies somewhere in between these two limiting cases.

INTRODUCTION

SEVERAL years ago a basic physical study of acoustic coupling devices was undertaken with a view toward gaining a better understanding of the performance of such systems. It was hoped that with better understanding the performance of these devices, of which the well-known horn is an example, could be improved upon.

Somewhat earlier Professor H. E. Hartig suggested that the tractrix curve looked promising as an acoustical horn structure. The tractrix, however, does not lie among the horn contours predicted by the Webster plane wave theory.[1,2] The plane wave theory seems plausible provided the horn does not flare too quickly. The flare of the tractrix, on the other hand, varies along its length, being exponential at the throat and finally becoming infinite at the mouth. A formulation of acoustic wave propagation in the tractrix horn, therefore, does not follow strictly the Webster theory although one of the requirements is that they become identical at distances far removed from the mouth, i.e., at the throat of the horn. The problem suggested was an investigation of the transition of sound energy from a plane wave form at the horn throat to a spherically symmetric form at the mouth. It was hypothesized that one could approximate this ideal transition by following the curve of a tractrix.

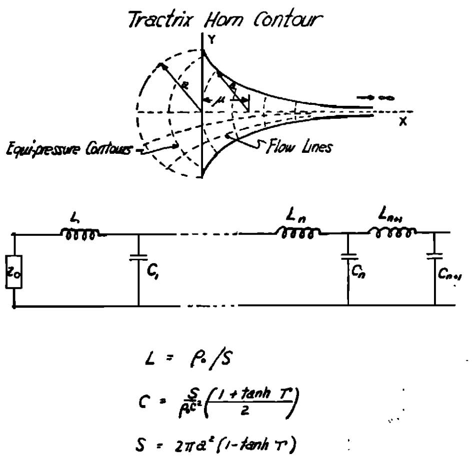

FIG. 1. Geometry of the tractrix horn and analogous electrical transmission line.

In the following a formulation of the propagation on the basis of a one-parameter spherical wave front theory is discussed. It is not the purpose of this study to develop a more generalized horn theory but rather to present other avenues of approach to horn design and to develop techniques for handling the propagation problems encountered.

PROPAGATION THEORY

The geometry of the tractrix horn is shown in Fig. 1, where the horn structure is taken as a figure of revolution about the X or principal axis. The tractrix coordinate μ represents the distance from the origin of coordinates to the intercept of the generating arm with the principal axis. An equation for the tractrix curve in the XY plane is written in parametric form as

$$X=\mu-a\tanh\mu/a$$
$$Y=a\operatorname{sech}\mu/a, \tag{1}$$

where a is the length of the generating arm.

A one-parameter formulation of the propagation theory proceeds on the assumption that the flow lines are tractrixes with a generating arm of length a and the surfaces of constant phase are spherical sectors of constant radii a. Hence, a is also the radius of the horn mouth and forms a convenient design parameter. Moreover, the equiphase contours are orthogonal to the flow lines and may be represented in the XY plane

* Presented at the forty-sixth meeting of the Acoustical Society of America, Cleveland, October 15, 1953.
† This work was supported by a grant-in-aid of research from the Graduate School of the University of Minnesota.
‡ On leave of absence to the Massachusetts Institute of Technology during the academic year 1954–1955.
[1] A. G. Webster, Proc. Natl. Acad. Sci. 5, 275 (1919).
[2] V. Salmon, J. Acoust. Soc. Am. 17, 212 (1946).

by the family of circles,

$$(\mu - X)^2 + Y^2 = a^2. \tag{2}$$

In the acoustic horn, Fig. 1, these spherical equipressure contours, visualized as moving from right to left, then represent outgoing waves. Thus, wave motion may be expressed in terms of one-parameter μ under steady-state excitation.

It is convenient to visualize acoustic wave propagation in horns as analogous to electromagnetic wave propagation along a nonuniform transmission line. Since acoustic energy losses along the horn are small, it is expedient to neglect the added complications caused by dissipation either to the side walls or in the gas. It turns out for the low range of frequencies, Fig. 2, and for the horn dimensions used in these studies that reduction in axial response due to boundary-layer viscosity and heat conduction losses at the side walls is less than one-half decibel. This estimate is based upon loss perturbation calculations similar to those discussed[3] by the writer involving attenuation in uniform tubes. Hence, neglecting losses, the schematic representation of the analogous transmission line is shown in Fig. 1, where the series acoustic inductance per length L and the shunt acoustic capacity per unit length C can be calculated from the physics and geometry and expressed as

$$\begin{aligned} L &= \rho_0/S \\ C &= S(1+\tanh\tau)/2\rho_0 c^2, \end{aligned} \tag{3}$$

where $S = 2\pi a^2(1-\tanh\tau)$ is the sector area, and $\tau = \mu/a$ is a dimensionless variable. In the above ρ_0 is the mean density of the gas and c is the velocity of sound in a "free-field." Consequently, expressions for the equations of motion for harmonic excitation of angular frequency ω and under the added assumption of negligible losses take the form

$$\begin{aligned} dp/d\tau &= -j\omega a L U, \\ dU/d\tau &= -j\omega a C p, \end{aligned} \tag{4}$$

where p the excess acoustic pressure and U the volume velocity are both assumed constant over S. The dissipationless one-parameter wave equation for the tractrix horn is calculated from Eq. (4) and expressed in the form

$$p'' + (S'/S)p' + k^2a^2\left(\frac{1+\tanh\tau}{2}\right)p = 0, \tag{5}$$

where $k = \omega/c$, and the prime indicates differentiation with respect to the variable τ.

One can obtain the essential information needed for our calculations by looking for traveling wave solutions to Eq. (5). In the interest of expressing these solutions in closed form one resorts to wave perturbation calculations. Thus, further approximations must be made. While rigor is certainly sacrificed, comparisons between theoretical impedance calculations and experimental measurements show that the approximations can be made quite good. They are at least accurate enough to predict general performance characteristics. A better idea as to the exactness of the perturbation calculations is obtained by comparing them with exact solutions to the Ricatti impedance equation for the tractrix horn obtained by analog computer techniques. It turns out that the wave perturbation approximations become poorer the lower the ratio of horn mouth dimensions to wavelength of the radiation.

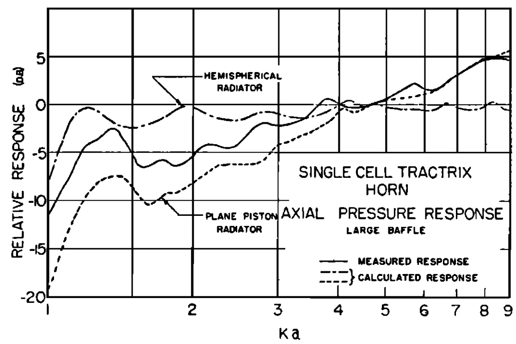

FIG. 2. Comparison between measured and calculated axial response.

In the usual manner we seek solutions to Eq. (5) of the form

$$p = W/S^{\frac{1}{2}} = A(\tau)\epsilon^{\pm j\theta(\tau)}/S^{\frac{1}{2}}, \tag{6a}$$

where

$$W'' + K^2 W = 0 \tag{6b}$$

and

$$K^2 = k^2a^2\left(\frac{1+\tanh\tau}{2}\right) - (S'/2S)^2 - \frac{d}{d\tau}(S'/2S). \tag{6c}$$

In Eq. (6a) the minus and plus signs indicate wave motion in the positive and negative τ directions, respectively.

If Eq. (6a) is to be a solution to Eq. (5), then the amplitude A and the phase θ must satisfy the following relations[4]

$$\theta''/\theta' + 2A'/A = 0 \tag{7a}$$

and

$$\theta'^2 = K^2 + A''/A. \tag{7b}$$

Equation (7a) is integrable in closed form, and its solution may be expressed as

$$A^2\theta' = K_1(ka), \tag{8a}$$

where K_1 is a constant of integration. The value of this constant is calculated from asymptotic solutions to Eq. (5), i.e., for distances far removed from the mouth of the horn. It turns out that the asymptotic form of Eq. (5) is identical with the Webster equation for an

[3] R. F. Lambert, J. Acoust. Soc. Am. 25, 1068 (1953).

[4] V. Salmon, J. Acoust. Soc. Am. 17, 199 (1946).

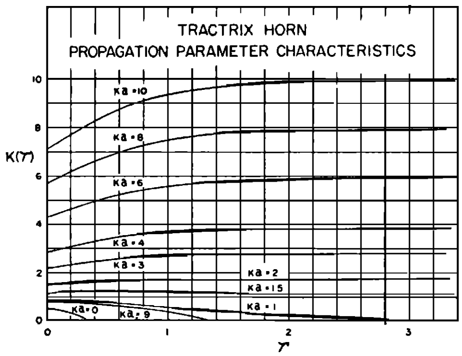

FIG. 3. Characteristics of the propagation parameter $K(\tau)$ for various values of Ka.

exponential horn of flare constant $1/a$. Solutions[5] to the asymptotic equation are well known. One calculates for the tractrix horn

$$A^2\theta' = (k^2a^2-1)^{\frac{1}{2}}. \tag{8b}$$

However, solutions to Eq. (7b) are not so easy to find. The complexity of the calculation depends upon the analytical form of K, where A and θ' are related by Eqs. (8a), (8b). In the present case the expression for K, Eq. (6c), has a limiting value of $[(k^2a^2+\frac{1}{2})/2]^{\frac{1}{2}}$ at $\tau=0$ and approaches $(k^2a^2-1)^{\frac{1}{2}}$ as τ approaches infinity, Fig. 3. We note that even at high frequencies, i.e., $k^2a^2 \gg 1$, K is not a slowly varying function of τ.

At this point in the analysis one may resort to any one of several wave perturbation methods. The choice is largely a matter of convenience. Probably the best known are the WKB[6] and the Liouville[7] methods.

A first approximation to the phase constant θ' can be obtained by examining Eq. (7b) and imposing the condition that

$$\int_0^b A''/A d\tau = \int_0^b \bar{\theta}'^2 d\tau - \int_0^b K^2 d\tau = 0, \tag{9}$$

where b is the normalized length of the horn. This condition yields

$$\bar{\theta}' = \left[\frac{1}{b}\int_0^b K^2 d\tau\right]^{\frac{1}{2}} \tag{10}$$

as an approximate value for the phase constant. For a tractrix horn of length $b=3.25$, one calculates

$$\bar{\theta}' = [.852(k^2a^2-1)+0.168]^{\frac{1}{2}}. \tag{11}$$

The cutoff parameter of the horn $\bar{k}_c a$ is calculated from the condition $\bar{\theta}'=0$. One calculates from Eq. (11) that $\bar{k}_c a = 0.895$, a value to be compared with a theoretical cutoff parameter $k_c a = 1$ for the asymptotic exponential horn.

To the order of approximation used in deriving Eq. (10) the amplitude A, Eqs. (8a), (8b), is constant. However, comparisons between throat impedance values calculated from this first-order solution and impedance values obtained from an analog computer solution to the Ricatti impedance equation indicate that the approximation, $A=$constant, is not very accurate. Let us formulate the Ricatti equation for the tractrix horn and see just what considerations are involved.

The impedance looking toward the mouth end of the horn is given by the ratio

$$Z = p/U = j(ka\rho_0 c/S)p/p'. \tag{12}$$

By substituting Eq. (12) into Eq. (5) and after some manipulation one arrives at the first-order differential equation

$$z' - (S'/S)z - j(ka/2\rho_0 c^2)(S'/S)z^2 - jka\rho_0 c = 0, \tag{13}$$

where $z=ZS$ is the specific acoustic impedance. Equation (13) is the Ricatti equation for the tractrix horn whose general solution may be written[4] in terms of A and θ in the form

$$z = R + jX = ka\rho_0 c\left[\theta' \coth(j\theta+\delta) - j\left(\frac{S'}{2S} - \frac{A'}{A}\right)\right]^{-1}, \tag{14}$$

where the reflection constant δ is evaluated at the mouth of the horn from the relation

$$\delta = \coth^{-1}\left\{\frac{1}{\theta'(0)}\left[ka\rho_0 c/z(0) + j\left(\frac{S'}{2S} - \frac{A'}{A}\right)_{\tau=0}\right]\right\} \tag{15}$$

and $z(0)$ is the impedance loading the horn. The propagation parameters A and θ appearing in Eqs. (14) and (15) must satisfy Eq. (7) for all τ.

In an effort to obtain better agreement as between the theoretical and Reeves analog computer (Reac) calculations, the writer endeavored to obtain more accurate perturbation solutions to Eq. (7b). It seems reasonable that a first approximations to θ' can be had by simply setting $\theta'=K$. One then recalculates θ' and A to any degree of approximation desired by iteration. Thus, to a first approximation

$$\theta' = K; \tag{16a}$$

$$A'/A = -\tfrac{1}{2}K'/K. \tag{16b}$$

By calculating A''/A from Eqs. (16a), (16b) and substituting this result into Eq. (7), one obtains a second approximation to the propagation constants,

[5] P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., New York, 1948), pp. 265–85.
[6] J. C. Slater and N. H. Frank, *Introduction to Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1933), p. 346.
[7] S. A. Schelkunoff and M. C. Gray, Bell System Tech. J. 27, 350 (1948).

namely,

$$\theta' = \left[K^2 + \frac{3}{4}\left(\frac{K'}{K}\right)^2 - \frac{1}{2}\frac{K''}{K}\right]^{\frac{1}{2}} \qquad (17a)$$

and

$$\frac{A'}{A} = -\frac{1}{2}\frac{\theta''}{\theta'}. \qquad (17b)$$

Throat impedance characteristics calculated from Eqs. (14), (15), (17a), (17b), (19a), and (19b) are shown in Fig. 4. The agreement between the Reac calculations and the perturbation calculations will be noted. No attempt was made to carry the perturbation to a higher degree of approximation because of the difficulty involved in calculating the phase θ.

A comparison between propagation measurements and wave perturbation calculations based upon the foregoing theory is discussed in the following section.

DISCUSSION

In order to obtain performance data on the tractrix horn and to check the validity of the one-parameter spherical wave front theory, several horn structures were tested under steady state excitation and for several baffle mountings. The details of acoustical measurements on single and multicell horn structures are described in a companion paper.[8]

To facilitate theoretical calculations, axial response measurements were taken with the horn mounted in a very large baffle. For all practical purposes the baffle may be regarded as being infinite. Preliminary measurements were made to establish the certainty of the 6-db reduction in rms pressure amplitude per doubling of distance. It turned out that the geometry was satisfactory for frequencies above 200 cps.

After examing both axial and polar response data on the single cell structure it was concluded that neither hemispherical nor plane piston radiation laws would fit the experimental pattern satisfactorily. However, in almost any event these distributions probably represent limiting cases. Consequently, it is instructive to compare semiempirical radiation calculations with the actual measurements. In all response measurements constant current was applied to the voice coil terminals of the driver unit.

In Fig. 2, axial response characteristics calculated for hemispherical and plane piston radiation patterns are compared with the measured data. These data are plotted *versus* the frequency-geometry parameter $ka = 2\pi a/\lambda$, where λ is the wavelength of the radiation. The theoretical response is calculated by equating output power at the horn mouth to input power at the throat neglecting losses. The input power is calculated under the added assumption of constant throat velocity and using measured values of throat impedance, Fig. 4. The data in Fig. 2 are corrected for the response of the W. E. 640A microphone and its associated electronic equipment as well as the response of the W. E. 555 driver unit. This comparison of the data suggests that the horn radiation response lies somewhere in between these two limiting cases.

In another endeavor to gain a better understanding of the coupling at the horn mouth and to check the accuracy of the wave perturbation calculations, throat impedance measurements were taken in the frequency range ka equals 1 to 10. In Fig. 4(a), theoretical throat impedance values calculated on the assumption of hemispherical radiation are plotted together with experimental measurements. Theoretical calculations are shown for both wave perturbation and Reac solutions to the Ricatti equation. In Fig. 4(b) the same comparisons are made for a plane piston radiation impedance.

The Ricatti equation was adapted to the Reac by substituting $z = z_1 + jz_2$ into Eq. (13) and separating into real imaginary parts as follows:

$$z_1' = -(1+\tanh\tau)(1 - ka/\rho_0 c z_2)z_1 \qquad (18a)$$

$$z_2' = -(1+\tanh\tau)[z_2 + ka/2\rho_0 c(z_1^2 - z_2^2)] + ka\rho_0 c. \qquad (18b)$$

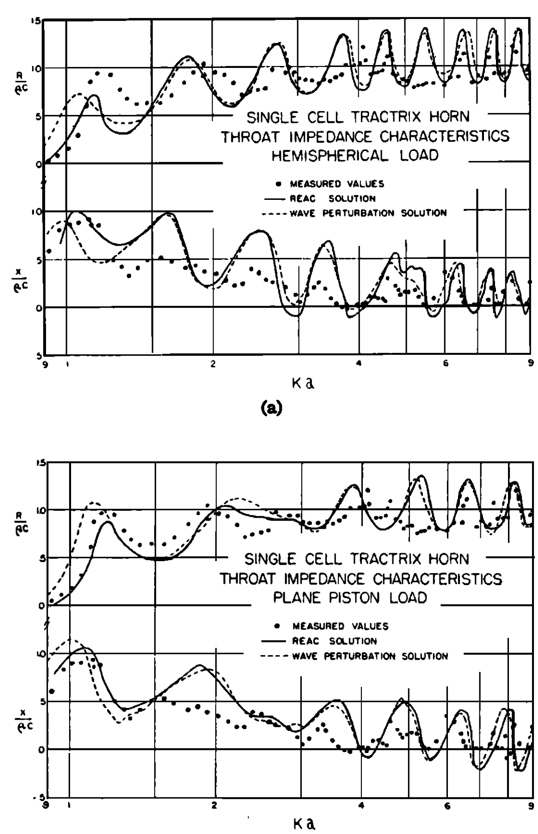

FIG. 4. Comparisons between measured and calculated throat impedance.

[8] A. O. Jensen and R. F. Lambert (following paper), J. Acoust. Soc. Am. 26, 1029 (1954).

These equations were then solved simultaneously on the computer.

Measurements of input impedance at the throat end of the horn, Fig. 4, reveal the presence of reflected waves. These reflections imply impedance mismatches at the mouth giving rise to resonances in the horn structure. One designs a horn to eliminate as far as possible serious mismatch over the useful frequency range. The question now arises as to the nature of the impedance loading the horn. An exact calculation of horn radiation impedance even for simpler geometrics is a problem which has never been solved. It was decided after a few preliminary runs on the Reac to limit the choices of $z(0)$ to a hemispherical cap and an equivalent plane piston load. The frequency characteristic of the radiation impedance for these two limiting cases is well known and expressions for $z(0)$ may be written[9] as

$$z_h(0)=\rho_0 c/(1+1/jka) \tag{19a}$$

and

$$z_p(0)=\rho_0 c\{[1-2J_1(2ka)/2ka]+j2K_1(2ka)/(2ka)^2\} \tag{19b}$$

for the hemispherical cap and plane piston loads, respectively.

The general agreement as between the Reac solution and the measurements will be noted. The hemispherical load, Fig. 4(a), seems to predict about the correct number of resonances. However, agreement at the lower frequencies is not as good as in the case of a plane piston load, Fig. 4(b). In Fig. 4(b) the over-all agreement seems to be somewhat better, although there do exist discrepancies over finite ranges at higher frequencies.

The agreement as between the Reac and wave perturbation calculations is also to be noted. These theoretical calculations are for all practical purposes identical save for frequencies in the immediate vicinity of cutoff. Such results develop confidence in the soundness of the techniques outlined above for handling propagation problems and perhaps may point the way toward better understanding and better design.

[9] Reference 5, pp. 332–33.

SUMMARY

Experimental measurements of axial pressure response and throat impedance are presented and compared with theoretical calculations neglecting losses for a single cell tractrix structure over the frequency range ka equals 1 to 10. These comparisons reveal that the horn radiation pattern lies somewhere in between a hemispherical and a plane piston distribution. Throat impedance measurements also agree reasonably well with impedance values calculated from a one-parameter formulation of the Ricatti impedance equation over the same frequency range. Both comparisons, however, suggest some departure from the simple radiation patterns. It should be noted that the polar response data[8] do show a pronounced columniation of the sound energy along the principal axis of the horn as the frequency is raised, a result which seems to be characteristic of radiation from a flexible diaphragm.

Attempts[8] to improve the off-axis radiation response by employing multicellular horn structures met with some success. While the uniformity of the angular distribution is for the most part improved, some sacrifice in smoothness in the axial frequency response results.

Throat impedance measurements on the single-cell structure indicate the presence of reflected waves in the horn which cause frequency response fluctuations. However, axial response measurements[8] over the audio-frequency range reveal a relatively smooth response indicating that the reflections are not serious.

We conclude from this study that the experimental findings substantiate the one-parameter spherical wave formulation of the propagation over a limited frequency range, say from ka equals 1 to 10. Taking into consideration the complicated geometry and the simplifying assumptions and approximations necessary to formulate a tenable theory, it is indeed gratifying that the general characteristics can be predicted as closely as these comparisons reveal.

In conclusion the writer expresses sincere appreciation to Professor H. E. Hartig for his encouragement and gratefully acknowledges helpful discussions with Professor E. L. Hill. Finally, the writer extends grateful thanks to Mr. P. N. Hess of the University of Minnesota's Computing Center for his aid in numerical work on the Reeves analog computer.