Reservoir Capacitance Required for Low-Frequency Sine Signals from a Class AB Audio Power Amplifier with an Uncontrolled Single-Phase Rectifier Power Supply with Capacitor Input

T.P. Gootee, 26 March 2013

This is a work in progress. This revision corrects the method used for the case where the signal frequency is greater than the AC Mains frequency, and predicts capacitance values that are verified, by comparison with a detailed LT-Spice simulations, to be at the threshold of clipping for a given rated maximum RMS output power.

This revision also recognizes that the presented method for the case of signal frequency less than mains frequency *uses an incorrect assumption* and underestimates the required capacitance for most signal frequencies below the AC Mains frequency. A subsequent revision will present the correct method for that case.

In any case, the last section, covering the constant-DC peak-equivalent case, does provide a good upper-bound capacitance that will be sufficient for any signal shape and any frequency, within the rated maximum output power.

The ideal capacitor's differential equation is:

$$i(t) = C\frac{dv}{dt}$$

or

$$\frac{dv}{dt} = \frac{1}{C}i(t)$$

where dv/dt is the time rate of change of the voltage across the capacitor and i(t) is the current flowing into the designated-positive-voltage lead of the capacitor at any time t >= 0.

We can integrate both sides of that equation, in order to have v(t) instead of dv/dt:

(9): $$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

where v(t) is the voltage across the capacitor at any time t (>= 0), and the integral must be understood to be evaluated from 0 to t (because this forum doesn't enable me to put the limits by the integral sign).

Re-arranging a little, we get

$$v(t) - v(0) = \frac{1}{C}\int_0^t i(t)dt$$

which can be expressed using "delta" notation, as

(10): $$\Delta v(t) = \frac{1}{C}\int_0^t i(t)dt$$

where Δv(t) is the change in the voltage across the capacitor due to the capacitor current i(t) existing since t = 0, at any time t >= 0.

Since the capacitor's ESR is a simple resistance, which adds a positive component to the capacitor's voltage for positive current i(t) into the positive-designated end of a capacitor, we can include the ESR terms in the Δv equation and validate them "by inspection":

(11): $$\Delta v(t) = \frac{1}{C}\int_0^t i(t)dt + ESR \cdot i(t) - ESR \cdot i(0)$$

That equation for Δv should be valid for any real-world current signal, i(t), at every time t >= 0.

Therefore, we could select i(t) = a·sin(ωt), noting that the integral with respect to time of a·sin(ωt)

= (-a/ω)·cos(ωt).

Substituting i(t) = a·sin(ωt) into equation (11), we have

(12): $$\Delta v(t) = \frac{1}{C}\int_0^t a\sin(\omega t)\, dt + ESR \cdot a\, sin(\omega t) - ESR \cdot a\, sin(0)$$

Rather than evaluating time only from 0 to t, we could use any interval, say t_0 to t_1.

CASE 1: ($f_s \geq f_{mains}$)

For the case of a capacitor with rectifier charging current pulses, and assuming that one charging pulse can fully-recharge the capacitor (which is not necessarily true), then if the time between charging pulses is greater than one-half of the period of a·sin(ωt), then (12) could be evaluated for an entire half period as a worst case for a single power rail in a class AB amplifier's power supply.

Equation (12) will be evaluated for that case, i.e. $f_s \geq f_{mains}$, with

$$\boldsymbol{a = i_{pk_{max}} = \frac{V_{pk_{max}}}{R_L} = \frac{\sqrt{2P_{RMS_{max}}R_L}}{R_L}}$$

f_s = the sine signal's frequency

Also, ESR was approximated as

ESR = 0.02 / (C · V_R),

where V_R = the Maximum DC Voltage Rating of the reservoir capacitor(s).

And to satisfy the non-clipping condition, the ripple amplitude, Δv, is constrained by:

$$\boldsymbol{\Delta v \; < \; V_{rail} - V_{clip} - V_{pk_{max}}}$$

With equation (12), integrating and evaluating from 0 to t gives:

$$\boldsymbol{\Delta v(t) \; = \; \left(-\frac{a}{\omega C}\right) \; cos(\omega t) - \left(-\frac{a}{\omega C}\right) cos(0) \; + \; ESR \cdot a \cdot sin(\omega t)}$$

and we get

$$\boldsymbol{\Delta v(t) \; = \; \left(\frac{a}{\omega C}\right)(1 - \cos(\omega t)) + ESR \cdot a \cdot \sin(\omega t)}$$

where "a" is the peak value of the sine current from capacitance C, ω = 2πf_s, and ESR is the capacitance's equivalent series resistance.

$$\boldsymbol{\Delta v(t) = \left(\frac{i_{pk}}{\omega C}\right) - \left(\frac{i_{pk}}{\omega C}\right) cos(\omega t) + \left(i_{pk} \cdot ESR\right) sin(\omega t)}$$

Previously, that equation was evaluated for ωt = π and it was assumed that the value obtained was the extremum of Δv. While that would usually give a reasonably-close estimate, when the ESR is significant the error of that approach would be significant.

In order to find the true maximum Δv (i.e. minimum rail voltage) under all conditions, we need to differentiate Δv(t), set the derivative to 0 (i.e. the point(s) where the slope changes from positive to negative; the extrema), and solve for the time(s) = t (or angle) when the peak(s) occurs(), and then evaluate the original equation to get the maximum Δv.

Differentiating:

$$\frac{d(\Delta v)}{dt} = \frac{d}{dt}\left(\left(\frac{-i_{pk}}{\boldsymbol{\omega C}}\right)\cos(\boldsymbol{\omega t})\right) + \frac{d}{dt}\left(\left(\boldsymbol{i_{pk} \cdot ESR}\right) \boldsymbol{sin(\omega t)}\right)$$

But, using The Chain Rule gives:

$$\frac{d}{dt}(\cos \boldsymbol{\omega t}) = \left(\frac{d}{d(\boldsymbol{\omega t})}\cos(\boldsymbol{\omega t})\right)\left(\frac{d}{dt}(\boldsymbol{\omega t})\right) = -\omega \sin(\boldsymbol{\omega t})$$

So we get

$$\frac{d(\Delta v)}{dt} = \left(\frac{i_{pk}}{\omega C}\right)(\omega \sin(\boldsymbol{\omega t})) + ESR \cdot \boldsymbol{i_{pk}} \cdot \boldsymbol{\omega}\,\mathbf{cos}(\boldsymbol{\omega t})$$

Setting the derivative of $\Delta v(t)$ to zero and solving for t, to find when the peak occurs:

$$0 = \left(\frac{i_{pk}}{C}\right)\sin(\boldsymbol{\omega t}) + ESR \cdot \boldsymbol{i_{pk}} \cdot \boldsymbol{\omega}\,\mathbf{cos}(\boldsymbol{\omega t})$$

$$\left(\frac{1}{C}\right) sin(\boldsymbol{\omega t}) = -ESR \cdot \boldsymbol{\omega}\, \boldsymbol{cos}(\boldsymbol{\omega t})$$

$$\frac{sin(\boldsymbol{\omega t})}{\boldsymbol{cos}(\boldsymbol{\omega t})} = -\boldsymbol{\omega} \cdot \boldsymbol{ESR} \cdot \boldsymbol{C}$$

$$\tan(\boldsymbol{\omega t}) = -\, \boldsymbol{\omega} \cdot \boldsymbol{ESR} \cdot \boldsymbol{C}$$

$$\boldsymbol{\omega t} = \boldsymbol{\pi} - \mathbf{tan^{-1}}(\boldsymbol{\omega} \cdot \boldsymbol{ESR} \cdot \boldsymbol{C})$$

$$\boldsymbol{t} = \frac{\boldsymbol{\pi} - \mathbf{tan^{-1}}(\boldsymbol{\omega} \cdot \boldsymbol{ESR} \cdot \boldsymbol{C})}{\boldsymbol{\omega}}$$

Substituting an approximation for ESR, in terms of capacitance value and voltage rating:

$$ESR \cong \frac{0.02}{C \cdot V_R}$$

gives

$$\boldsymbol{t} = \frac{\boldsymbol{\pi} - \mathbf{tan^{-1}}\left(\frac{\mathbf{0.02}\boldsymbol{\omega}}{\boldsymbol{V_R}}\right)}{\boldsymbol{\omega}}$$

$$\boldsymbol{t} = \frac{\boldsymbol{\pi} - \mathbf{tan^{-1}}\left(\frac{\mathbf{0.04}\,\boldsymbol{\pi}\,\boldsymbol{f_s}}{\boldsymbol{V_R}}\right)}{\boldsymbol{\omega}}$$

Substituting that expression for t into the original equation for $\Delta v(t)$ gives the maximum Δv as:

$$\boldsymbol{\Delta v_{max}} = \left(\frac{i_{pk}}{\omega C}\right)\left(1-\cos\left(\omega\left[\frac{\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)}{\omega}\right]\right)\right) + ESR \cdot i_{pk} \cdot \sin\left(\omega\left[\frac{\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)}{\omega}\right]\right)$$

$$\Delta v_{max} = \left(\frac{i_{pk}}{\omega C}\right)\left(1-\cos\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)\right) + \left(\frac{0.02}{C \cdot V_R} \cdot i_{pk}\right) \cdot \sin\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)$$

Solving for the capacitance, C:

$$C = \left(\frac{i_{pk}}{\Delta v_{max}}\right)\left[\left(\frac{1}{\omega}\right)\left(1-\cos\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)\right) + \left(\frac{0.02}{V_R}\right)\left(\sin\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)\right)\right]$$

Imposing the clipping-threshold conditions by substituting for Δv:

$$\Delta v_{max} < V_{rail} - V_{clip} - V_{pk_{max}}$$

$$C = \left(\frac{i_{pk}}{V_{rail} - V_{clip} - V_{pk_{max}}}\right)\left[\left(\frac{1}{\omega}\right)\left(1-\cos\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)\right) + \left(\frac{0.02}{V_R}\right)\left(\sin\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\,f_s}{V_R}\right)\right)\right)\right]$$

Substituting an expression for ipk in terms of the rated maximum output power:

$$i_{pk_{max=}} \frac{\sqrt{2P_{rated}R_L}}{R_L}$$

The final result is:

(13):

$$\boldsymbol{C \geq \left(\frac{\sqrt{2P_{rated}R_L}}{R_L\left(V_{rail} - V_{clip} - \sqrt{2P_{rated}R_L}\right)}\right)\left[\left(\frac{1}{2\pi f_s}\right)\left(1 - cos\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\, f_s}{V_R}\right)\right)\right) + \left(\frac{0.02}{V_R}\right)\left(sin\left(\pi - tan^{-1}\left(\frac{0.04\,\pi\, f_s}{V_R}\right)\right)\right)\right]}$$

for $f_s \geq f_{mains}$.

The previous approximate equation is given for reference:

$$\boldsymbol{C \; > \; \frac{\left(\frac{\sqrt{2P_{rated}R_L}}{R_L}\right)\left(\frac{0.02(2\pi f_s)}{V_R}+1\right)}{(\pi f_s)\left(V_{rail}-V_{clip}-\sqrt{2P_{rated}R_L}\right)}}$$

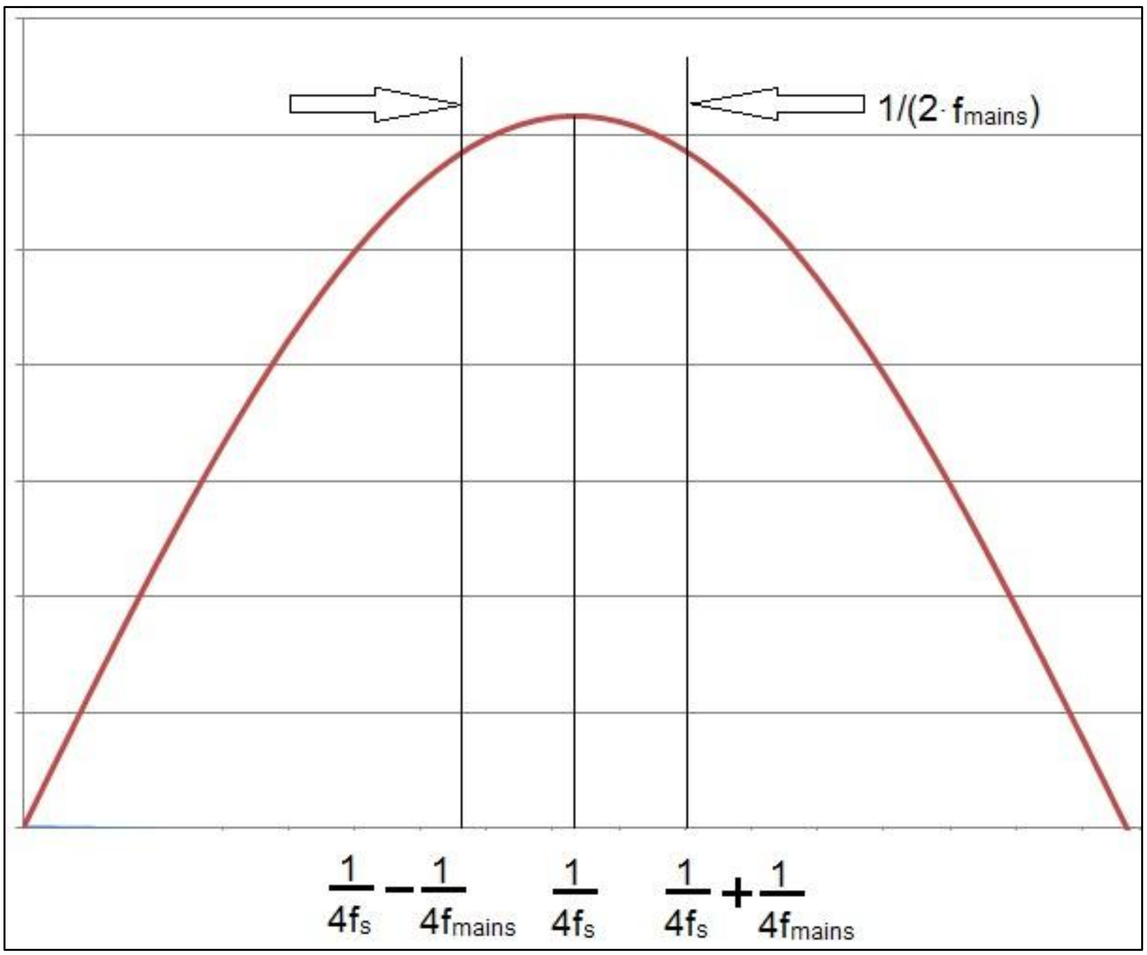
for

$\boldsymbol{f_s \geq f_{mains}}$.

CASE 2: ($f_s < f_{mains}$)

NOTE that this case is acknowledged to be incorrectly analyzed, in this revision. The assumption, below, that the maximum Δv excursion would occur for a charging pulse centered on the signal peak is correct, as far as it goes. But that does not necessarily (or even usually) give the maximum total Δv. This method will be modified in a future revision.

For the case where the sine's frequency is less than the AC Mains frequency, the charging pulses, occurring at twice the mains frequency, would have a period that was shorter than one half cycle of the signal. In that case, the worst-case Δv would occur for the worst-case total current drawn from the capacitor, which would be when the charging pulses were on opposite sides of, and equidistant from, the peak of the sine current waveform that would be drawn from the capacitor, as depicted in the figure:



Equation (12) was evaluated from t_0 to t_1, with

f_s = signal frequency,

$t_0 = (1 / (4 \cdot f_s)) - (1 / (4 \cdot f_{mains}))$,

$t_1 = (1 / (4 \cdot f_s)) + (1 / (4 \cdot f_{mains}))$, and

$a = i_{pk} = V_{pk} / R_L = \text{sqrt}(2 \cdot Pwr_{max} \cdot R_L) / R_L$

Also, ESR was approximated as

$ESR = 0.02 / (C \cdot V_R)$,

where V_R = the Voltage Rating of the reservoir capacitor(s).

And to satisfy the non-clipping condition, the ripple amplitude, Δv, is constrained by:

$\Delta v < V_{rail} - V_{clip} - V_{pk\_max}$.

Setting up equation (12) with the details discussed, from the figure:

$$\Delta v = \frac{1}{C}\int_{t_0=\frac{1}{4f_s}-\frac{1}{4f_{mains}}}^{t_1=\frac{1}{4f_s}+\frac{1}{4f_{mains}}} \left(\frac{v_{pk_{max}}}{R_L}\right)\sin(\omega t)\,dt + ESR\left(\frac{v_{pk_{max}}}{R_L}\right)sin(\omega t_1) - ESR\left(\frac{v_{pk_{max}}}{R_L}\right)sin(\omega t_0)$$

After performing the integration, we have:

$$\Delta v = \left(-\frac{v_{pk_{max}}}{\omega R_L C}\right)\left[\cos\left(\omega\left(\frac{1}{4f_s}+\frac{1}{4f_{mains}}\right)\right) - cos\left(\omega\left(\frac{1}{4f_s}-\frac{1}{4f_{mains}}\right)\right)\right] + \left(\frac{ESR \cdot v_{pk_{max}}}{R_L}\right)\left[\sin\left(\omega\left(\frac{1}{4f_s}+\frac{1}{4f_{mains}}\right)\right) - \sin\left(\omega\left(\frac{1}{4f_s}-\frac{1}{4f_{mains}}\right)\right)\right]$$

But

$$\omega\left(\frac{1}{4f_s}+\frac{1}{4f_{mains}}\right) = \frac{2\pi f_s}{4f_s} + \frac{2\pi f_s}{4f_{mains}} = \frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)$$

giving

$$\Delta v = \left(\frac{v_{pk_{max}}}{R_L}\right)\left\{-\frac{1}{\omega C}\left[cos\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - cos\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right] + ESR\left[sin\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - sin\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right]\right\}$$

Using the approximation

$$\boldsymbol{ESR} = \frac{\mathbf{0.02}}{\boldsymbol{C \cdot V_R}}$$

gives

$$\Delta v = \left(\frac{v_{pk_{max}}}{R_L \cdot C}\right)\left\{-\frac{1}{\omega}\left[\cos\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - \cos\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right] + \frac{0.02}{V_R}\left[\sin\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - \sin\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right]\right\}$$

Solving for C:

$$C = \left(\frac{v_{pk_{max}}}{R_L \cdot \Delta v}\right)\left\{\left(-\frac{1}{\omega}\right)\left[\cos\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - \cos\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right] + \left(\frac{0.02}{V_R}\right)\left[\sin\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) - \sin\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right]\right\}$$

Substituting

$\mathbf{\Delta v < V_{rail} - V_{clip} - V_{pk\_max}}$

and

$\boldsymbol{\omega = 2\pi f_s}$

and

$V_{pk_{max}} = \sqrt{2P_{rated}R_L}$

gives **(14):**

$$C > \left(\frac{\sqrt{2P_{rated}R_L}}{R_L\left(V_{rail}-V_{clip}-\sqrt{2P_{rated}R_L}\right)}\right)\left[\left(-\frac{1}{2\pi f_s}\right)\cos\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) + \left(\frac{0.02}{V_R}\right)\sin\left(\frac{\pi}{2}\left(1+\frac{f_s}{f_{mains}}\right)\right) + \left(\frac{1}{2\pi f_s}\right)\cos\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right) + \left(-\frac{0.02}{V_R}\right)\sin\left(\frac{\pi}{2}\left(1-\frac{f_s}{f_{mains}}\right)\right)\right]$$

, for $\mathbf{f_s < f_{mains},}$

where

V_{rail} = the fully-loaded power rail voltage across the capacitance,

V_{clip} = the amplifier's minimum (i.e. "clipping") voltage, per the datasheet,

V_{pk_max} = the peak sine output voltage at maximum rated power, and

V_R = the capacitors' rated maximum DC voltage.

With the equation above, as the signal frequency approaches zero, the indicated minimum capacitance value approaches the value given by the equation in the next section, which is reasonable because, at very low frequencies, the top of the sine waveform would appear almost flat over the interval between charging pulses, much like DC at the sine's peak level.

Reservoir Capacitance Required for Worst-Case Signal

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The worst-case current draw from the reservoir capacitor(s) would occur for a constant DC output voltage equal to the peak sine voltage that occurs when the rated maximum power is delivered to the load.

The output power would then be the square root of 2 times (1.414x) the rated maximum output power. But since the amplifier is capable of producing that peak output voltage, it is reasonable to assume that it could occur for a somewhat longer period of time, in the worst case. Even just a very low frequency sine could be a close approximation, for one or more charging intervals.

In that case, and using the same ESR approximation and clipping constraint as above, the following equations result, for the minimum capacitance required to avoid clipping with ANY signal with voltages that do not exceed the rated maximum-power sine's peak level.

Starting with (9), and assuming that the current, between charging pulses, is a constant DC current from the capacitor to the load, we can add an ESR term to account for the constant voltage drop across the capacitor's ESR, for the constant DC current:

$$\Delta\boldsymbol{v}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{C}}\int_{t_0}^{t_1} \boldsymbol{i}(\boldsymbol{t})\,\boldsymbol{dt} + \boldsymbol{ESR}\cdot\boldsymbol{i}(\boldsymbol{t})$$

Where i(t) has been defined to be negative when flowing out of the capacitor's positive terminal, which would make both the of the terms on the right side of the equation negative, as expected.

Between charging pulses, i(t) is a constant DC current with amplitude equal to the peak sine current that would occur at the rated maximum power output:

$$\boldsymbol{i}(\boldsymbol{t}) = \boldsymbol{i}_{\boldsymbol{pk}_{\boldsymbol{max}}}$$

So we have

$$\Delta\boldsymbol{v}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{C}}\int_{t_0}^{t_1} \boldsymbol{i}_{pk_{max}}\,\boldsymbol{dt} + \boldsymbol{ESR}\cdot\boldsymbol{i}_{pk_{max}}$$

$$\Delta\boldsymbol{v}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{C}}\left(\boldsymbol{i}_{pk_{max}}\boldsymbol{t}_1 - \boldsymbol{i}_{pk_{max}}\boldsymbol{t}_0\right) + \boldsymbol{ESR}\cdot\boldsymbol{i}_{pk_{max}}$$

$$\Delta\boldsymbol{v}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{C}}\Big(\boldsymbol{i}_{pk_{max}}(\boldsymbol{t}_1 - \boldsymbol{t}_0)\Big) + \boldsymbol{ESR}\cdot\boldsymbol{i}_{pk_{max}}$$

But the discharge time will not be longer than the time between charging pulses. So we have

$$(\boldsymbol{t}_1 - \boldsymbol{t}_0) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{f}_{mains}}$$

$$\Delta v = \frac{1}{C}\left(\frac{i_{pk_{max}}}{2f_{mains}}\right) + ESR \cdot i_{pk_{max}}$$

Approximating ESR as

$$ESR = \frac{0.02}{C \cdot V_R}$$

gives

$$\Delta v = \frac{i_{pk_{max}}}{2f_{mains}C} + \frac{0.02 \cdot i_{pk_{max}}}{C \cdot V_R}$$

$$\Delta v = \left(\frac{i_{pk_{max}}}{C}\right)\left(\frac{1}{2f_{mains}} + \frac{0.02}{V_R}\right)$$

$$C = \left(\frac{i_{pk_{max}}}{\Delta v}\right)\left(\frac{1}{2f_{mains}} + \frac{0.02}{V_R}\right)$$

But

$$i_{pk_{max}} = \frac{V_{pk_{max}}}{R_L} = \frac{\sqrt{2P_{rated}R_L}}{R_L}$$

and the ripple voltage, Δv, must also satisfy

$$\Delta v < V_{rail} - V_{clip} - V_{pk_{max}}$$

So we get:

(15):

$$C > \left(\frac{V_{pk_{max}}}{R_L\left(V_{rail} - V_{clip} - V_{pk_{max}}\right)}\right)\left(\frac{1}{2f_{mains}} + \frac{0.02}{V_R}\right)$$

where

$$V_{pk_{max}} = \sqrt{2P_{rated}R_L}$$

A similar equation was also derived, for a possible lower-bound estimate of the minimum required capacitance, with V_{RMS_max} substituted for V_{pk_max} in the numerator, which is equivalent to assuming a constant DC output voltage, at the max rated (RMS) output level.