Analysis of an Uncontrolled Single-Phase Power Supply Rectifier Circuit with Source Inductance and Resistance, a Reservoir Capacitor with ESR, and a Constant-Current Load

Revision dV6

02 DEC 2012, Thomas P. Gootee

(Draft copy. Not yet reviewed.)

Introduction:

Several authors have provided analyses of AC-to-DC power supply rectifier circuits. But apparently none has included all of the components of a practical circuit of the type that might be used with an audio amplifier or a voltage regulator as the load, while also not using coarse approximations that make them inaccurate when the ripple voltage is not small compared to the average DC output level. The effects of source inductance, which can cause the output voltage peaks to be higher than the input voltage, have usually also not been included.

This paper's analysis includes the inductance and resistance of a power transformer or other source, which enables the effects of transformer size, e.g. Volt-Amp Rating and Output Voltage Rating, to be included, and also enables the output voltage peaks and troughs to be more-accurately predicted.

A reservoir/filter capacitance is also included, with the capacitance's ESR (Equivalent Series Resistance), which improves the accuracy of the results and enables a designer to better-specify a real capacitor and also to better-evaluate the trade-offs involved when considering using multiple smaller parallel capacitances.

Also, rather than a simple resistive load, a power amplifier or voltage regulator in series with a resistive load is assumed. This not only makes the analysis more useful. It also adds a degree of simplification, because in order to enable the determination of the minimum reservoir capacitance value that will guarantee that the power rail voltage cannot violate the amplifier or regulator's "dropout" region, a constant DC or square wave signal (in the case of an amplifier load), *with an "on" voltage equal to the peak sine voltage for a given output power rating*, is a convenient quasi-worst-case signal for analysis. And considering only a portion of the positive part of one output cycle with a square wave (or constant DC) signal is equivalent to considering a constant current load.

In the case of an amplifier load with a resistive load in series, it is assumed that the amplifier behaves as a time-varying resistance, which maintains the output current through its driven load resistance at a constant level during the “on” portions of a square wave (or constant DC) signal, within its ability per its PSRR (Power Supply Rejection Ratio), which is easily verified to be the case, with a Spice simulation, where it can also be seen that the minima of the power supply’s output voltage ripple must not fall below the sum of the voltages across the load resistance and the amplifier’s power output stage, which for the purposes of this analysis might, for example, consist of a bipolar transistor’s collector-emitter voltage in series with a sub-one-Ohm resistor.

A voltage regulator load has a very similar requirement for the power supply output voltage minima to stay above the region where its dropout voltage clearance would be violated.

Attempting a closed-form mathematical analysis of simple rectifier circuits is surprisingly complex and tedious and most authors resort to using one or more approximations, in order to simplify or make possible the analysis. One common approximation technique that is used is to assume that the filter capacitance discharges linearly, rather than exponentially. In this paper’s analysis, the use of a constant-current load makes that approximation unnecessary, while still providing the benefit of that type of simplification. Another common assumption, that the ripple voltage amplitude is very small compared to the average output voltage, will be avoided, here.

The circuit equations herein will necessarily include the voltage, current, and resistance of one or more rectifier diodes. Since this analysis is intended to be implemented in computer software, if necessary it will be assumed that if given the diode voltage or current, then the diode’s voltage, current, and resistance can all be known, either using a table-lookup function or an equation.

Typically, when attempting to derive a closed-form solution for these types of circuits, a significant problem is encountered when, inevitably, the point in time (or phase angle) when the decaying exponential capacitor voltage intersects the rising sinusoidal transformer secondary output voltage must be found, to determine when the rectifier diodes begin conducting again. The resulting equation is a transcendental equation, which has no closed-form mathematical solution.

Some authors have used clever methods to circumvent that problem, such as using the first few terms of a Taylor Series approximation for the sinusoid or the exponential or both. And some authors have used the LambertW function, we^w, to advantage. But since this paper will focus on defining the relevant differential equations and implementing a numerical solution, the transcendental equation will be avoided and, in addition to the differential equations that describe the circuit behaviors, only equations that define the rectifier mode-transition conditions will need to be derived. And those equations will even be allowed to contain differential terms, since all of the derivatives will be available, at all times during the numerical solver software’s iterations.

Variables and Symbols Used:

t	time (seconds)
f	AC Mains frequency (Hertz)
ω	AC Mains angular frequency (= $2\pi f$ Radians per Second)
T	AC Mains period (= $1/f$ Seconds)
t_{on}	time when diode conduction starts (Seconds)
t_{off}	time when diode conduction stops (Seconds)
t_c	tcharge: time interval during which diode conducts and capacitor charges
t_d	tdischarge: time interval during which capacitor discharges
$V_{out}(t)$	Output voltage at time t (Volts)
$V_{cap}(t)$	Capacitor voltage at time t (Volts)
$Vs_{in}(t)$	Transformer secondary's input voltage at time t
$R_d(t)$	Resistance of diode at time t
$i_d(t)$	Current from transformer secondary winding at time t, when diode is conducting (Amps)
i_L	Constant current through load and amplifier output stage
C	Capacitance value of power supply's reservoir capacitance (Farads)
$i_c(t)$	Current through capacitor at time t
R_E	ESR (Equivalent Series Resistance) of filter capacitance (Ohms)
L_s	Leakage inductance of transformer secondary, or source inductance (Henrys)
$V_{Ls}(t)$	Voltage across inductance L_s at time t
R_s	Resistance of transformer secondary, or source (Ohms)
$V_{Rs}(t)$	Voltage across resistance R_s at time t
$V_E(t)$	Voltage across capacitor's equivalent series resistance, R_E, at time t

Analysis:

This analysis will mainly consider the power supply behavior during the positive "on" portion of a constant DC or square wave output signal. Only the positive voltage rail of a dual-polarity power supply supplying a single active load with series load resistance will be considered. The equivalent of a push-pull type of amplifier load, driving a single active load with series load resistance, drawing a constant current during the portions of the constant DC or square wave output being considered, will be assumed.

Circuit operation:

When the transformer secondary's output voltage amplitude is high-enough, relative to the output voltage, the rectifier diode(s) will be forward biased and will allow current to flow from the transformer secondary to the filter capacitor and the load, as needed. When the transformer secondary's output voltage amplitude is not high-enough, the rectifier diode(s) will not be forward biased and no current will flow from or to the transformer secondary through the rectifier diode(s) and any needed load current will then be supplied by only the reservoir capacitor.

At a minimum, we desire to be able to accurately predict the power supply output voltage's and current's maximum, minimum, and average steady-state values, given the circuit parameters.

One possible version of the type of circuit being considered is shown in Figure 1. (Note that the current source labeled "iload" is not strictly an ideal current source, as might be assumed from the figure, since it contains an active load and that active circuit's driven load resistance. Thus the depicted current source can have resistance and can have a non-zero voltage across it.)

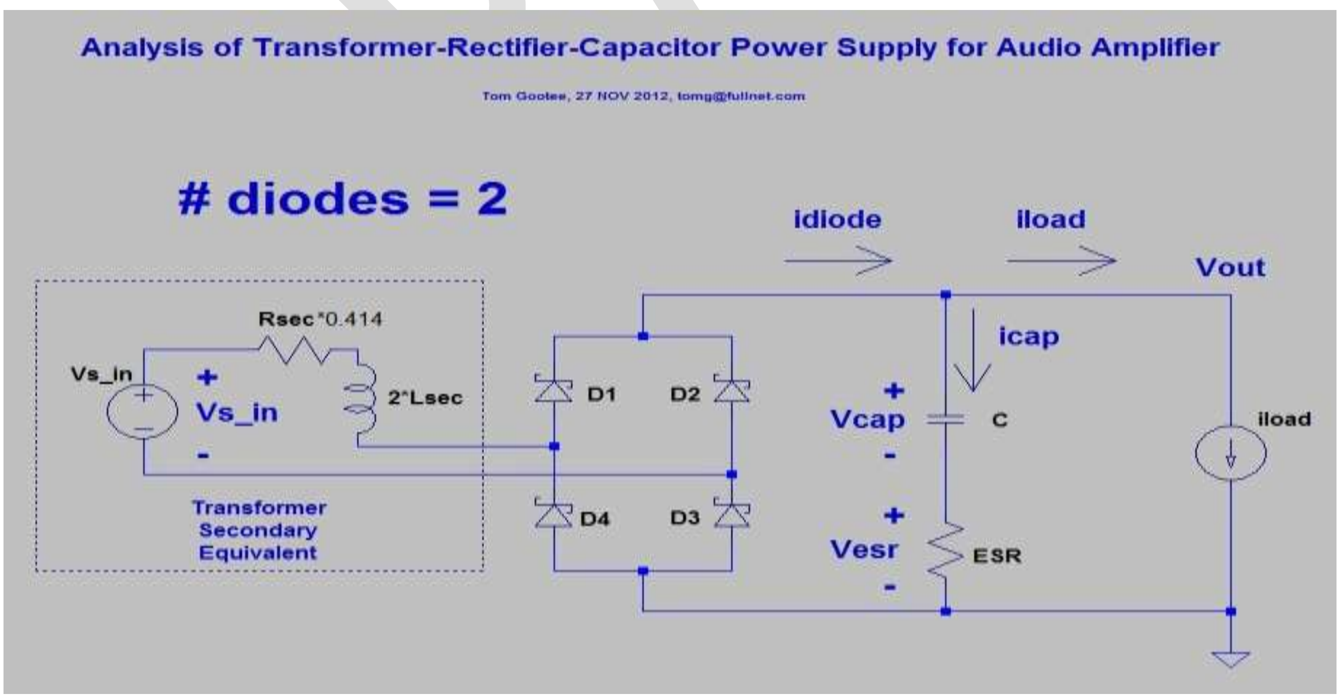

Figure 1

By inspecting the circuit of Figure 1 and using Kirchoff's Current Law, we can write

$$\boldsymbol{i_d(t) \ = i_C(t) + i_L} \quad \textbf{(1)}$$

Also, we can see that

$$V_{out}(t) \ = \ V_{cap}(t) + V_E(t)$$

$$\boldsymbol{V_{out}(t) \ = \ V_{cap}(t) + i_C(t) \cdot R_E} \quad \textbf{(1a)}$$

Using the equation that relates the current and voltage of an ideal capacitor,

$$\mathrm{C} \cdot \frac{\mathrm{dV_{cap}(t)}}{\mathrm{dt}} = i_c(t)$$

and substituting into (1a), we could also use, equivalently:

$$\boldsymbol{V_{out}(t) \ = V_{cap}(t) + R_E \cdot C \cdot \frac{dV_{cap}(t)}{dt}} \quad \textbf{(2)}$$

Diodes OFF:

During the portion (if any) of each half of the AC cycle when none of the rectifier diodes are conducting, the capacitance, its ESR, and the constant-current load can be considered, alone:

When the diodes are not conducting,

$$\mathrm{i_d(t) = 0}$$

$$i_C(t) \ = \ -i_L$$

$$\boldsymbol{C \cdot \frac{dV_{cap}(t)}{dt} = -i_L} \quad \textbf{(3)}$$

$$\boldsymbol{\frac{dV_{cap}(t)}{dt} = -\frac{1}{C} i_L} \quad \textbf{(4)}$$

Equation (5) is a first-order differential equation with a known solution for $V_{cap}(t)$ as a function of time, for any initial capacitor voltage, $V_{cap}(0)$.

We can also see that V_{out} can be expressed in terms of only V_{cap} and constants:

$$V_{out}(t) \ = \ V_{cap}(t) - i_L \cdot R_E$$

Diodes ON:

When the diodes are conducting,

$i_d(t) = i_C(t) + i_L$, which is the same as

$$\boldsymbol{i_d(t) = C\frac{dV_{cap}(t)}{dt} + i_L} \quad \textbf{(5)}$$

And therefore

$$\boldsymbol{\frac{di_d(t)}{dt} = C\frac{d^2V_{cap}(t)}{dt^2}} \quad \textbf{(6)}$$

Noting that the equation relating voltage and current in an ideal inductor is

$$\mathrm{V = L\frac{di(t)}{dt}}$$

we can use Kirchoff's Voltage Law and equation (6) and sum the voltages around the loop to zero, assuming the input voltage for the transformer's secondary winding is $Vs_{in} \cdot \cos(\omega t + \varphi)$:

$$Vs_{in} \cdot cos(\omega t + \varphi) - \left(C\frac{dV_{cap}(t)}{dt} + i_L\right)R_s - L_sC\left(\frac{d^2V_{cap}(t)}{dt^2}\right) - \left(C\frac{dV_{cap}(t)}{dt} + i_L\right)R_d = V_{cap}(t) + R_EC\frac{dV_{cap}(t)}{dt}$$

Regrouping and rearranging, we get

$$\boldsymbol{\frac{d^2V_{cap}(t)}{dt^2} + \left(\frac{R_s+R_d+R_E}{L_s}\right)\frac{dV_{cap}(t)}{dt} + \frac{V_{cap}(t)}{L_sC} = \left(\frac{Vs_{in}}{L_sC}\right)cos(\omega t + \varphi) - \left(\frac{R_s+R_d}{L_sC}\right)i_L} \quad \textbf{(7)}$$

which is a standard form for a non-homogeneous second-order ordinary differential equation that is typical of driven series RLC circuits.

However, the diode resistance, R_d, is not constant, and is a non-linear function of the time-varying current through the diode or the voltage across the diode (we could choose either one).

Numerical Solution:

For implementation of the numerical solutions of equations (4) and (7), a particular diode was chosen and a p-Spice model of the diode, supplied by the diode's manufacturer, was used, in LT-Spice, to produce data points for the diode's current versus the voltage across the diode. The data points were exported from LT-Spice and MS Excel was then used to calculate the resistance for each pair of current and voltage points, using Ohm's Law. Then, curve-fitting software was used to determine an equation that approximated the diode's resistance, versus the current through the diode, which was valid over a sufficient current range of interest.

For an OnSemi MBR20100CT diode, one of the equations found to be sufficiently accurate is:

Y = (A+B*X**C) / (D+X**C) with

A=0.48953E+00   B=0.62604E-02   C=0.87411E+00   D=0.44516E-03

or

$$\boldsymbol{R_d(i_d) = \frac{0.48953+0.0062604\cdot i_d^{\,0.87411}}{0.00044516+i_d^{\,0.87411}}} \qquad \textbf{(8)}$$

The plots of R_d versus i_d from (8), and the plotted percent error, are given in Figure 2 and Figure 3. Note that the plot for the resistance versus the current was truncated so that resistances above ten Ohms are not shown, in order to enable viewing more detail for low resistances. In reality, equation (8) is valid for diode resistances up to at least 100 to 500 Ohms, which corresponds to a diode current of 2 mA to 0.2 mA, and the equation could probably be used for diode resistances up to 2500 Ohms, which corresponds to a diode current of about 0.5 μA.

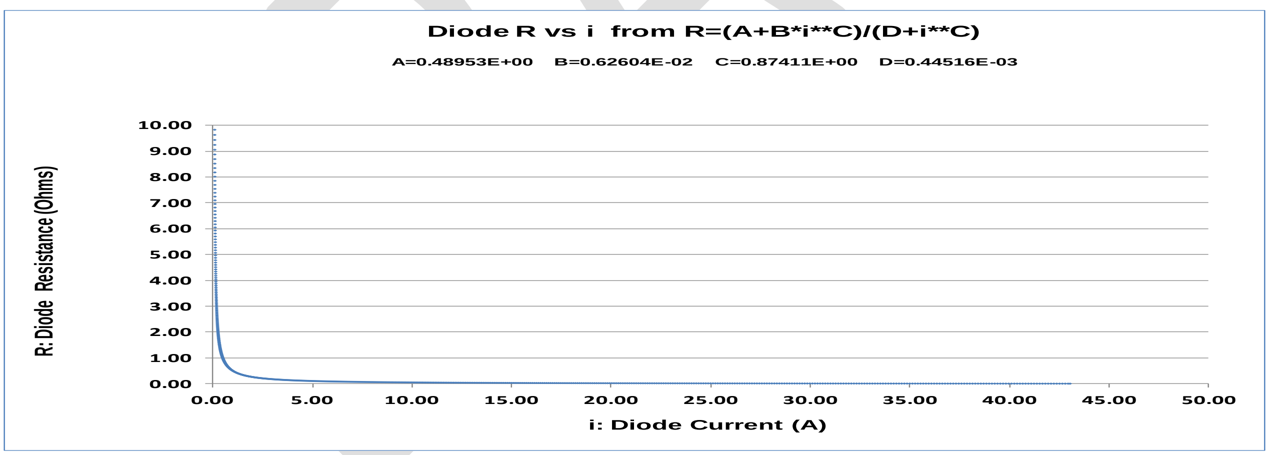

Figure 2

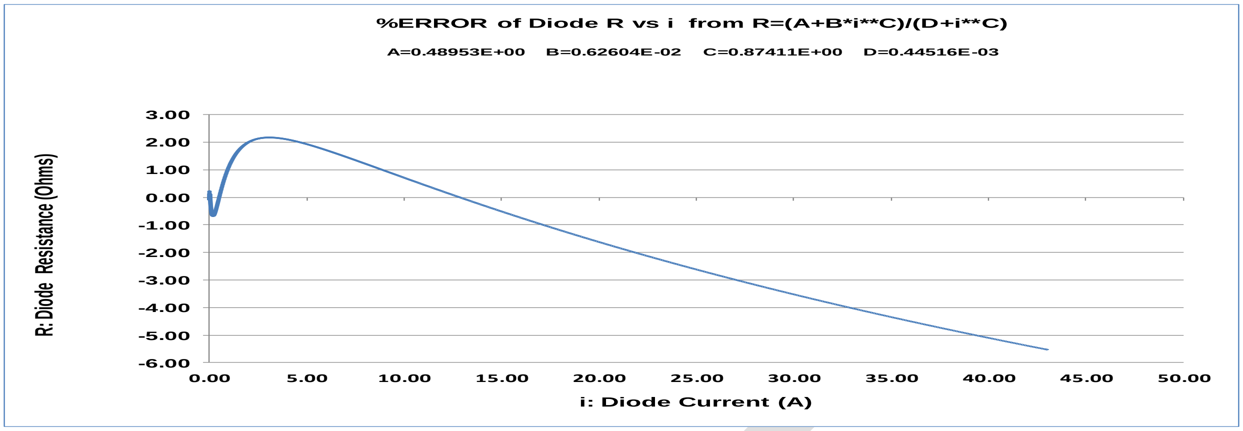

Figure 3

In order to implement a numerical algorithm to generate $V_{cap}(t)$ and related values, a fourth-order Runge-Kutta method was selected, which is capable of solving two simultaneous first-order differential equations. In order to implement the algorithm, equation (7) was converted from a second-order equation into a system of two first-order equations, where

$$\boldsymbol{y = V_{cap}(t)} \quad \textbf{(9)}$$

$$\boldsymbol{z = \frac{dV_{cap}(t)}{dt}} \quad \textbf{(10)}$$

$$\mathbf{g = z} \quad \textbf{(11)}$$

$$\boldsymbol{f = \frac{dz}{dt}} \quad \textbf{(12)}$$

Substituting (9)-(12) into equation (7), while allowing for ndiodes in series instead of just one, gives:

$$\boldsymbol{f = \left|\left(\frac{Vs_{in}}{L_s C}\right) cos(\omega t + \varphi)\right| - \left(\frac{R_s + ndiodes \cdot R_d(i_d(t))}{L_s C}\right) i_L - \left(\frac{R_s + ndiodes \cdot R_d(i_d(t)) + R_E}{L_s}\right) z - \frac{y}{L_s C}}$$
(13)

where $R_d(i_d(t))$ is the diode resistance, from equation (8), a non-linear function of the time-varying diode current, and the absolute value of the input voltage is used in order to simulate the behavior of having both pairs of diodes, more easily, by having only one pair and one equation.

Some background information: (simplified to the point of being slightly incorrect)

Numerical "solution" of differential equations is conceptually simple. Often, differential equations arise because we know, from physical laws, for example, how to write mathematical expressions and equations involving the derivatives (rates of change) of variables of interest, but we then desire an equation for the behavior of the variable itself, as a function of time for example, in the particular situation for which the differential equation is applicable. The desired mathematical equation for the variable of interest (as a function of time, in our case) IS "the solution" of whatever differential equation we have written.

There are several formal methods for solving linear differential equations with time-invariant coefficients. But not much is known about solving most non-linear differential equations. Therefore, iterative numerical methods are often used, to simply "run" (i.e. simulate) the system described by the differential equation, or set of equations, in order to see how the system would behave.

The basic idea is that if small-enough time steps are chosen, and initial values of the variable of interest and some of its derivatives are chosen or calculated, or guessed-at, for the first several instants of time, *then the value of the variable at each "next" instant of time, from then on, can simply be calculated*, by calculating the derivatives and applying the coefficients in the differential equation(s), and the calculated values will be accurate-enough if the time steps used are small-enough.

After all, the variable's "first derivative" (e.g. dV/dt) is simply the slope of the plot of the values (e.g. of the waveform) of the variable itself, and we can use those values, from two adjacent time steps, and the time step length, to calculate that slope. The "second derivative" (e.g. d^2V/dt^2) is just the slope of the plot of the first derivative. So we can calculate its new value the same way, and so on for any higher-order derivatives. If we set the time steps to be small-enough, and calculate the values of the variable and each of its needed derivatives as we iterate forward to each new time step, and plug those values into the differential equation(s) at each step and find the new values of the variable and its derivatives for the next time step, the calculated values will follow the trajectory that the actual system would follow, and the resulting set of times and values will be a plot of the actual solution of the equation versus time, which in our case will be the waveform that would result from running the actual circuit!

It turns out that the obvious way of calculating the derivatives, using the simple slope formula, which is called Euler's Method, is not the best we can do. While it can almost always be made to work well-enough, Euler's Method has a built-in problem which can make it require excessively-small time steps if we want it to be accurate-enough as time gets farther from $t=0$. A slightly more-complicated but more-accurate and better-behaved method of calculating the new value (at the next time step) of a variable and its first derivative is the Runge-Kutta type of algorithm. It comes in more than one level of complexity but the fourth-order version is great for our purposes, here.

Back to the current problem:

We have equation (4) for the times when the diodes are not conducting and equation (7) for when the diodes are conducting. Therefore, the software that iterates through the time steps will need to keep track of which of the two equations it is currently using, and, it will have to calculate when to switch to the other equation.

Determination of when a "Diode Off → On" transition occurs:

By inspecting the circuit in Figure 1 we can see that the voltage at the upstream end of the diode is the transformer secondary's input voltage minus the voltages across the secondary's resistance and inductance. However, if the diodes are not conducting and we need to determine when they should start conducting again, then we can use the fact that the current through each diode has been zero for some time, and thus the current through the secondary winding and its resistance and inductance has also been zero for some time. Therefore the first derivative (rate of change) of the same current must also be zero, as we approach the diode turn-on time. And therefore the voltages across both the resistance and the inductance of the secondary are zero, in that situation. Therefore the voltage at the upstream end of the diode must be equal to the input voltage of the secondary winding.

We also know that when the diodes are not conducting, the capacitor is discharging linearly into the load, due to the constant load current pulling charge out of the capacitor. The voltage at the downstream end of the diode in the forward path is designated as the output voltage, Vout. We can now say that if the diode is not conducting then it will start conducting when:

$$|\boldsymbol{Vs_{in}\, cos(\omega t + \varphi)}| \geq \boldsymbol{V_{out}(t)} \quad \textbf{(14)}$$

Using equation (2), we get

$$|\boldsymbol{Vs_{in}\, cos(\omega t + \varphi)}| \geq \boldsymbol{V_{cap}(t) + R_E \cdot C \cdot \frac{dV_{cap}(t)}{dt}} \quad \textbf{(15)}$$

In terms of the numerical solution, we then have

$$|\boldsymbol{Vs_{in}\, \cos(\omega t + \varphi)}| \geq \boldsymbol{y + R_E \cdot C \cdot z} \quad \textbf{(16)}$$

Equation (16) must become true at the diode turn-on time.

Determination of when a "Diode On → Off" transition occurs:

In the case when the diode(s) is(are) conducting and the numerical algorithm software needs to determine when it(they) should "turn off":

We assume that a diode's current is zero, or goes to zero, at the moment the diode stops conducting.

But

$i_d(t) = i_C(t) + i_L$

So, when $i_d(t)$=0:

$0 = i_C(t) + i_L$

$i_C(t) \to -i_L$

$$C \cdot \frac{dV_{cap}(t)}{dt} \leq -i_L$$

$$\boldsymbol{C \cdot \frac{dV_{cap}(t)}{dt} \geq i_L}$$

In terms of the numerical solution, we then have:

$$\boldsymbol{C \cdot z \geq i_L} \quad \textbf{(17)}$$

Equation (17) must become true at the diode turn-off time.

Initial Conditions:

The differential equations specify the rate of change of the variable of interest, and those of its derivatives. But in order to simulate a specific solution, a starting point must be provided, for the variable's value and for each derivative except the highest-order one.

For the second-order differential equation that was derived for the circuit in Figure 1, the initial capacitor voltage and its first derivative must be supplied, in order to generate a particular solution.

The initial capacitor voltage can be chosen to be any voltage. But if we are only interested in seeing the steady-state behavior, i.e. after the circuit has been running for some time and the response has settled to a steady condition, then we would probably want to select an initial voltage that is near-enough to the final average output voltage, to minimize the number of time steps needed. If we want to see the startup behavior of the power supply, we could choose zero as the initial voltage.

The software implementation described below provides an initial guess for the capacitor voltage that is intended to be near the final steady-state average value. But a user-entry field is provided, so that the user can override the initial capacitor voltage value.

For the software implementation described below, a simple method was used for specifying the initial value of dV_{cap}/dt, by assuming that the diodes were not conducting, so that dV/dt is, from (4):

$$\boldsymbol{\frac{dV_{cap}(0)}{dt} = -\frac{1}{C} i_L} \quad \textbf{(18)}$$

While (18) will not always be strictly correct (e.g. if the diodes are initially conducting), in practice for the circuits being considered the numerical algorithm quickly forces the solution to follow the correct trajectory, anyway.

The formal method for selecting the initial conditions would be to use the applicable closed-form solution of equation (7), differentiating the solution and using the resulting equation to help determine dV/dt at t=0, given V(t) at t=0. The non-linear time-varying diode resistance might complicate that process. And since we have chosen to use a numerical solution method instead of attempting to derive a complete closed-form solution, and since experimentation with the resulting numerical solver software has shown that the results are almost identical when using a large negative value for dV/dt and when using zero, the initial conditions will probably be handled sufficiently-well by setting up the software to allow user override of any automatically-calculated initial conditions.

Software Implementation:

The MS Excel platform was used to implement the numerical algorithm and a user interface. An image of the main user interface is shown in Figure 4.

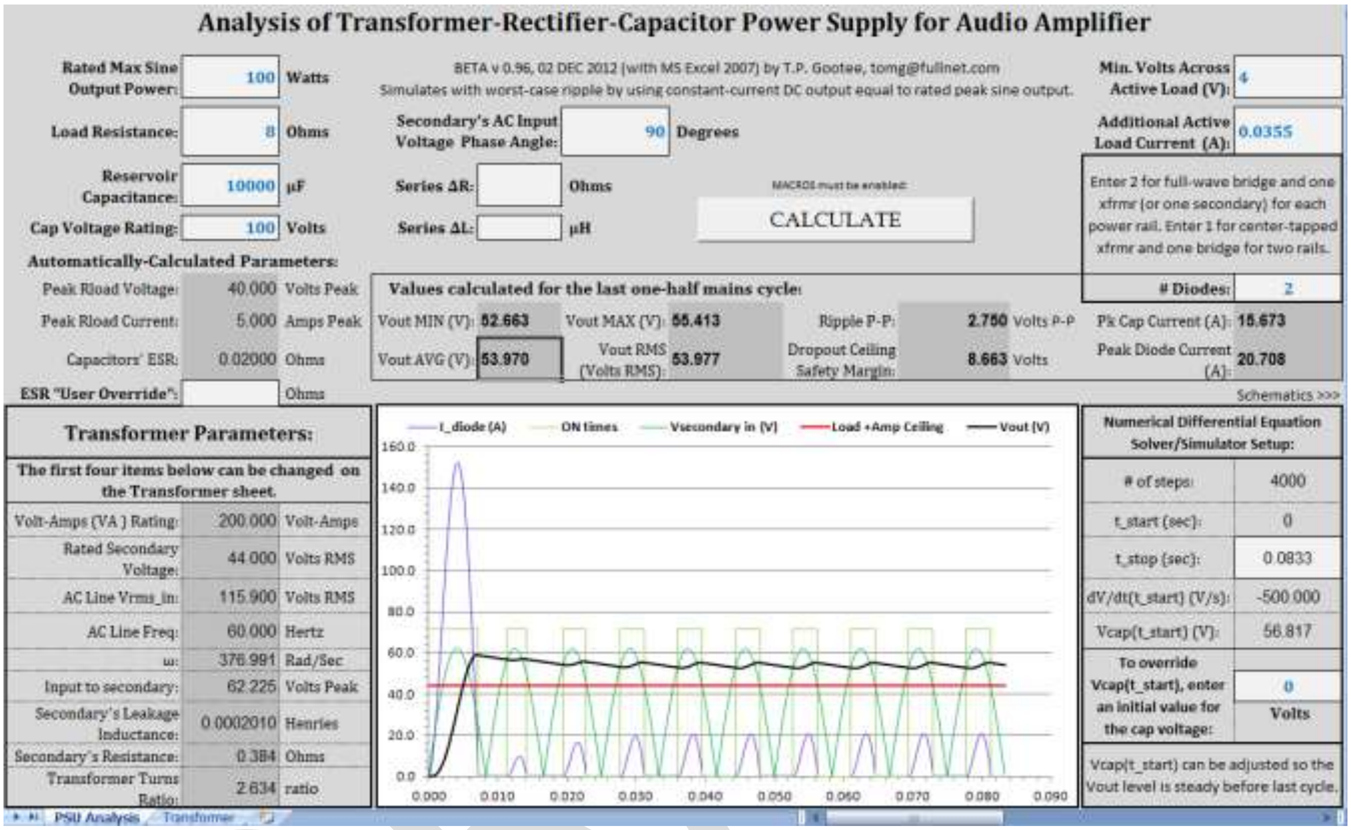

Figure 4 – Main user interface screen for numerical solver in MS Excel

A VBA (Visual Basic for Applications) subroutine macro was written, which runs each time the "CALCULATE" button is selected. The VBA subroutine reads and writes values from the cells of the spreadsheet depicted in Figure 4 and implements the numerical solution algorithm. The values for the plots shown in Figure 4 are also calculated in the VBA subroutine, and are written in spreadsheet rows below the rows shown in Figure 4, with one set of calculations per row. An example of a small portion of the output rows is shown in Figure 5.

Software Limitations:

In a case where the user has entered an initial capacitor voltage that is not close to the final average output voltage level of the power supply, such as when zero is entered in order to see the start-up behavior, it will often be the case that the capacitor voltage will not be in a steady state when the stop time occurs, especially when a large capacitor value has been entered. In that case, many of the performance measures that are calculated and displayed, which are calculated using the data for the last half-cycle of the AC line voltage, will not be representative of the power supply's actual performance.

The software does not currently check whether or not steady state has been reached and also does not calculate the possible error due to the time-step duration.

To mitigate the problem, the user could increase the value of the stop time, t_stop. But unless the value of the number of time steps was also increased, then the time duration of each step would increase, which would result in diminished accuracy, to some unknown extent. The number of steps could also be increased, to avoid diminishing the accuracy of the results. But then the user would need to manually edit the plot settings, to include the additional rows of data that would be generated.

t	vcap	dvcap/dt	Vout	id	non-zero = yes	Vs_in* cos(wt+p)	ic	Ceiling Voltage	Vd sum
time (sec)	Capacitor Voltage	Cap Volts per Second	Output Voltage	Diode Current	Diode Conducting?	Secondary Voltage	Capacitor Current	Peak Load + Amp Voltage	Vdiodes
0.0000000	0.0000	-500.0000	-0.1000	0.0355	71.5592	0.0000	-5.0000	44.0000	0.638315
0.0000208	-0.0104	-501.3777	-0.1107	0.0217	71.5592	0.4885	-5.0138	44.0000	0.597281
0.0000417	-0.0209	-500.1646	-0.1209	0.0339	71.5592	0.9770	-5.0016	44.0000	0.634275
0.0000625	-0.0312	-496.6735	-0.1306	0.0688	71.5592	1.4654	-4.9667	44.0000	0.696591
0.0000833	-0.0415	-490.9548	-0.1397	0.1260	71.5592	1.9538	-4.9095	44.0000	0.753807
0.0001041	-0.0517	-482.9931	-0.1483	0.2056	71.5592	2.4420	-4.8299	44.0000	0.803413
0.0001250	-0.0616	-472.7751	-0.1562	0.3077	71.5592	2.9300	-4.7278	44.0000	0.846867

Figure 5 – Examples of output data generated for each time step