Loudspeaker Phase Characteristics and Time Delay Distortion: Part 2.

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Fourier Integral concepts are explored for the relation existing between a function in the frequency domain and its time domain counterpart. A derivation is obtained for the effect of a loudspeaker's imperfect frequency response as a specific type of time delay distortion of the reproduced signal.

INTRODUCTION In an earlier paper [1] the definition of loudspeaker frequency response was expanded to include the phase of the pressure wave produced by an electrical stimulus as well as the conventionally measured amplitude. A technique of measurement was introduced which allowed a measurement to be made of this more complete response, and some examples were included of the response of common types of loudspeaker. Since the proper role of a loudspeaker is the acoustic reproduction of a time-dependent signal, the measurement of even the more complete frequency response is academic unless some inference can be obtained from this measurement as to whether the loudspeaker does its job well. Accordingly a presentation without proof was made of a means of visualizing the effect of imperfect loudspeaker frequency response as producing a time delay distortion equivalent to a frequency-dependent spatial distribution of otherwise perfect loudspeakers. It is the purpose of the present work to investigate the determination of temporal response from the more complete frequency response and develop this acoustic model.

In considering time response it must be remembered that engineers work in a causal world where cause distinctly precedes effect and time advances in its own inexorable fashion. No analysis performed on a network as complicated as a loudspeaker may be considered valid *if it violates* causality and allows the clock to run backward. Because of considerable mathematical complexity, the subject of time delay in a dispersive medium with absorption is generally avoided in most written material. The reader of such material is left instead with some simplified relations using the frequency phase spectrum, which for most systems yield time delay answers close to observed behavior. Those systems for which the answer violates a prior physical premise are considered anomalous. When all that is available on the frequency response of a loudspeaker is the pressure amplitude spectrum one cannot utilize the simplified temporal relations, and hence no questions arise. With the introduction of a means for measuring the complete frequency response one runs into immediate difficulty with application of the simplified concepts of time behavior because in many cases it is found that causality cannot be maintained.

In order to understand the distortion which a loudspeaker may impart to a time-dependent signal because of its imperfect frequency response, it becomes necessary to look more closely at the concept of frequency-dependent time delay and generate revisions required to present an understandable acoustic equivalent for an actual loudspeaker. This paper proceeds by first demonstrating why the common concept of group delay is not applicable to minimum-phase systems with absorption. Then a substitute for group delay is developed and is shown to provide the proper solution for some systems commonly considered to have anomalous behavior. Finally, a network concept is introduced which leads to an appropriate acoustic model for a loudspeaker.

GROUP DELAY, EXCESS DELAY, AND OVERALL TIME DELAY

Historically the concept of time delay in a dispersive medium was recognized as early as 1839 by Hamilton, but the distinction between phase delay and group delay seems to have been put on a firm foundation by Lord Rayleigh in publications in 1877 [2]. Rayleigh considered that group velocity represented the actual velocity of propagation of groups of energy in a medium. Group delay is defined to be the time delay in traversing a fixed distance at this group velocity [3]. To understand group delay one need only consider that the transformation from the analysis of a problem in the frequency domain to the solution in the time domain involves a Fourier Integral of the form

$$f(t)=(1/2\pi)\int_{-\infty}^{\infty} G(\omega)\, e^{\alpha(\omega)}\, e^{i[\omega t+\phi(\omega)]}\, d\omega \qquad (1)$$

This may fall into a class of integral equations of the type

$$f(t)=\int F(s)e^{tg(s)}ds \qquad (2)$$

where $s=\sigma+i\omega$, $g(s)=x+iy$ is an analytic function, t is large, positive, and real, and $F(s)$ varies slowly compared with the exponential factor [4, 5].

Lord Kelvin's method of stationary phase evaluates integrals of this type by deforming the path of integration where possible through saddle points where x is constant and

$$\partial x/\partial\sigma=\partial x/\partial\omega=0 \text{ and } \partial y/\partial\sigma=\partial y/\partial\omega=0 \qquad (3)$$

For this path the modulus of $\exp[t\cdot g(s)]$ is constant while the phase varies. When all of these conditions are met, not only may an asymptotic solution be achieved but what is more important, Eq. (3) shows that the major contribution to the integral takes place where the phase is stationary and

$$dg(\omega)/d\omega=0. \qquad (4)$$

When these conditions are applicable the major contribution to the solution of Eq. (1) occurs at a time t such that

$$t=-d\phi(\omega)/d\omega. \qquad (5)$$

Since time commences in the analysis at initiation of input stimulus, this means that the time delay of the signal through the network is this value of t, called group delay.

Since the principle of stationary phase is a commonly used derivation of the network theory concept of group delay, it is of utmost importance to note the restrictions on the use of this derivation. The most important restriction is that the modulus remain a slowly varying function of frequency in that region of the frequency domain where the phase is changing the least. This means that when working with a network element this condition may be met by solutions which involve very long time delays, such as transmission lines, or when applied to networks that have no amplitude variation with frequency, such as all-pass lattices where $\alpha(\omega)$ in Eq. (1) is always a constant.

In setting up relations for a network with absorption and short overall time delays, one gets an equation deceptively similar to Eq. (2) but with a substantial real as well as imaginary term in the exponent. The time function of Eq. (1) is an inversion integral evaluated along a path which is the entire imaginary axis from $-\infty$ through the origin to $+\infty$, closed to the left with a semicircle of infinite radius and the origin as center. This is done so as to encircle all singularities of the integrand for time greater than zero. The path of integration is restricted to the $i\omega$ axis when the expression of Eq. (1) is used and the real and imaginary parts of the exponent are related by the Cauchy-Riemann differential relations. Thus, even if $\alpha(\omega)$ is generally a slowly varying function, just at that point on the $i\omega$ axis where the phase is stationary, $\alpha(\omega)$ varies rapidly and one *cannot* use the principle of stationary phase. If the time delay of the network is small relative to several periods of the frequency under analysis, which is a condition commonly found in loudspeakers, then this inapplicability of stationary phase can lead to solutions for time delay which are absurd. Consider for example the circuit of Fig. 1. This network is certainly well behaved, yet the group delay is negative from zero frequency to the geometric mean of the transfer-function break points. Since obviously the output cannot predict the input, the only logical solution would be that the time delay of this network is not represented by group delay. There will of course exist a proper solution for time delay, but this requires a careful evaluation of the inversion integral through the saddle points of Eq. (3) where one may either use Kelvin's method of stationary phase with a path through the saddle points with x constant, or the method of steepest descent which chooses a path of integration so as to concentrate the large values of x in the shortest possible interval with y constant. The two methods are nearly equivalent, since the paths cross the same saddle points and can be deformed one into the other provided contributions from any singularities crossed are taken into account.

The minimum phase transfer function is the function with the minimum accumulation of phase lag (negative phase shift) as ω proceeds from dc to infinity. Because of this the accumulation of phase lag in a minimum phase

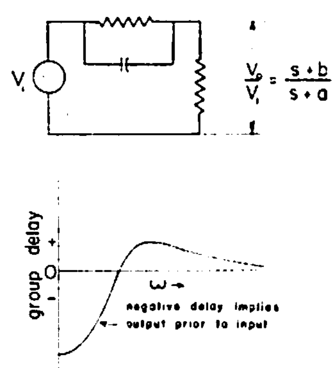


Fig. 1. A simple minimum phase circuit and its group delay, illustrating the inapplicability of group delay to such a causal circuit.

network may be negative as well as positive, as in the case of the circuit of Fig. 1. Since the Cauchy-Riemann equations actually define a minimum phase network and are necessary conditions that a circuit cannot predict the occurrence of a signal, the behavior of phase accumulation means that the group delay of Eq. (5) does *not* provide a measure of time lag in a minimum phase network.

While this might appear to destroy the concept of group delay for minimum phase networks, consider now the special non-minimum phase network called the allpass or flat network, with a constant amplitude of response [6, 7, 8]. For this network there is accumulated phase lag at a rate which is not negative at any frequency, yielding a group delay which is never negative. For this network, since $a(\omega)$ is constant, the principle of stationary phase is valid on the imaginary axis. The time delay thus calculated according to Eq. (5) is everywhere meaningful; this time delay of an allpass network with the transfer function

$$H(\omega) = e^{-i\theta(\omega)} \tag{6}$$

will be defined as excess delay

$$t_{\text{excess}} = d\theta(\omega)/d\omega. \tag{7}$$

One must be careful to observe that the excess delay is the time elapsed from the injection of a transient to the major contribution of the output waveshape. There may be minor ripples, or forerunners to use a phrase of Brillouin [9], which precede this major change as well as the latecomers which provide the effect commonly called ringing, but nonetheless the major change will occur at the time which was called excess delay.

If a network is minimum phase, there exists a unique relationship between amplitude and phase which allows a complete determination of phase from amplitude. If a network is non-minimum phase with a transfer function $H(\omega)$, there will exist a unique minimum phase network $G(\omega)$ with the same amplitude response, and an allpass network with a phase response $\theta(\omega)$ in cascade such that [10]

$$H(\omega) = G(\omega)e^{-i\theta(\omega)} = A(\omega)e^{-i\phi(\omega)}e^{-i\theta(\omega)}. \tag{8}$$

If the time delay characteristics of minimum phase networks and allpass lattices are considered, one can reconstruct the time behavior of any arbitrary physically realizable network. There will exist some total time delay of the network $H(\omega)$ which will be called t_{overall}. There will also exist some time delay for the minimum phase network $G(\omega)$ which will be called $t_{\text{min. phase}}$. The relation between these delays is

$$\begin{aligned} t_{\text{overall}} &= t_{\text{min. phase}} + t_{\text{excess}} \\ &= t_{\text{min. phase}} + [\partial\theta(\omega)/\partial\omega]. \end{aligned} \tag{9}$$

The commonly used group delay is the frequency slope of the total measured phase of $H(\omega)$, or from Eq. (5)

$$t_{\text{group}} = [\partial\phi(\omega)/\partial\omega] + [\partial\theta(\omega)/\partial\omega] \tag{10}$$

which may be expressed as

$$t_{\text{group}} = t_{\text{overall}} + \{[\partial\phi(\omega)/\partial\omega] - t_{\text{min. phase}}\}. \tag{11}$$

Consequently the group delay will be quite close to the overall time delay of the network if

$$t_{\text{overall}} >> \{[\partial\phi(\omega)/\partial\omega] - t_{\text{min. phase}}\}. \tag{12}$$

This is another verification that if a network has a sufficiently large overall time delay, then group delay may be considered a satisfactory substitute provided that the group delay of the equivalent minimum phase network is reasonably well behaved.

TIME DELAY AS A DISTRIBUTION

Turn now to a consideration of time delay in a general network. Attempts at a direct derivation of time delay do not seem particularly fruitful, since the classic definition requires that one make a sudden change in some parameter and see how long it takes before this change appears in the output; however, the moment a discontinuity is created in a time derivative of an electrical parameter, one no longer has that parameter but a large set of sideband frequencies which interfere with the measurement. Thus one is led to look for another solution which involves the relationship existing between frequency and time.

The relationship existing between a function in the time domain $f(t)$ and the same function in the frequency domain $F(\omega)$, is given by the Fourier integrals

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{13}$$

and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{14}$$

There is an obvious symmetry which analytically lets a function in time commute with a function in frequency. Indeed, if a function were given in a dummy parameter and one did not know whether it was of time or frequency, there would be no way of ascertaining the proper domain. A remarkable fact would arise if one blindly inserted this function into the wrong equation: if the function were as well behaved as any related to a real world containing dissipation, the answer would be correct in form. This is because functions may be transferred in the Fourier integral if the sign of one of the parameters is reversed [11]. The implications of this are enormous, as many facts laboriously proven in one domain may automatically be transferred to the other domain. For example, as pointed out in a previous paper [12], if one terminates a time series there exists a frequency overshoot analogous to Gibbs' phenomenon.

The commutation of parameters, then, gives the remarkable simplification that the analysis of a distribution in time due to a complex transfer function is isomorphic with the frequency distribution due to complex modulation in time. This isomorphism considerably frees our imagination when trying to cope with the concept of the time delay of a frequency. If one imagines that the variation of amplitude with frequency of a frequency transfer function is analogous to the variation of amplitude with time of a time transfer function, one can imagine that there are "time sidebands" analogous to the frequency sidebands of modulation theory. In the case of frequency, all values from $-\infty$ through zero to $+\infty$ are allowed and we conveniently identify negative frequency as a phase reversal of a positive frequency. For the parameter time it is conventional to start analysis for a value of zero and assume no activity prior to this.

This merely requires that the time function have an even and odd component which cancel each other for all times less than zero. For such a time function which does not allow prediction, this means that the frequency function similarly has an even (amplitude) and odd (phase) component, although these will not necessarily cancel out at any frequency. This physical realizability criterion also means that the frequency transfer function must have complex conjugate poles and zeros in order to satisfy the even-odd requirement.

The concept of time delay of a frequency component is not complete, since the functions discussed so far are voltages in terms of either frequency or time. Consider, however, a frequency function which has a distribution that is forming as we observe in real time. This is called the running transform $F_t(\omega)$ [13, 14]

$$F_t(\omega) = \int_{-\infty}^{t} f(t)e^{-i\omega t}dt. \tag{15}$$

In this case there is a distinct relation between the distribution of sideband energy and time. There will exist a spectral distribution of frequencies corresponding to an instant in time which may be single-valued, multiple-valued, or a continuous distribution. By interchanging time for frequency one may infer that the time delay of a network for a given frequency may also be a distribution. This goes a long way toward clarifying the confusion created by investigators who attempt to come up with a single-valued number for the delay of a network. In those regions in which the actual delay distribution is small or single-valued, the simple group delay scores very well, but in regions of moderate to large dispersion group delay falls down completely and even yields absurdities.

By observing the conjugate behavior of time and frequency it should be apparent to anyone familiar with modulation theory that a network frequency transfer function

$$F(\omega) = A(\omega)e^{-i\phi(\omega)} = e^{a(\omega)}e^{-i\phi(\omega)} \tag{16}$$

represents a distribution of time delayed functions around the value

$$t_{\text{group}} = [d\phi(\omega)/d\omega]. \tag{17}$$

Furthermore, the group delay will represent the absolute delay of each component only if

$$a(\omega) = \text{constant}. \tag{18}$$

The distribution around the group delay in Eq. (17) is certainly consonant with the paired echo concept of Wheeler and MacColl [15] which treats the effect of minor deviations from the ideal transfer function by expanding the time function around these deviations.

DELAY IN MINIMUM PHASE NETWORK

Having recognized that the true network time delay of Eq. (19) may not necessarily be single-valued and may even be a finite distribution, we turn out attention to deriving the form of a minimum phase time $t_{\text{min. phase}}$ for several simple expressions.

As shown earlier, the group delay of a network with constant gain is the proper delay. Consider the single

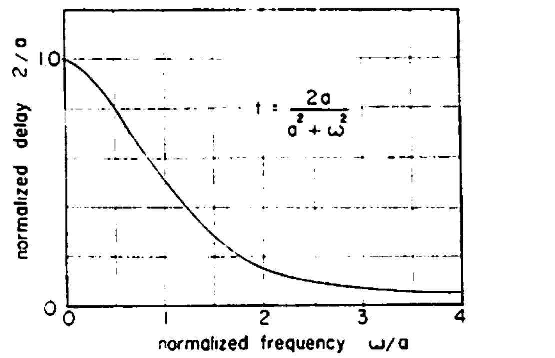


Fig. 2. Normalized plot of excess delay for a first-order allpass lattice.

pole allpass lattice function

$$L(\omega) = (s-a)/(s+a) = (i\omega - a)/(i\omega + a) \text{ for } \sigma = 0. \tag{19}$$

This is a constant gain function with a group delay

$$t_{\text{group}} = 2a/(a^2+\omega^2). \tag{20}$$

This is shown in Fig. 2. The time delay is maximum at zero frequency, and there is no delay at infinite frequency. There is also the very useful fact that single pole functions can be expressed as combinations of this lattice, for example,

$$\frac{1}{s+a} = \frac{1}{2a}\left(1 - \frac{s-a}{s+a}\right) \tag{21}$$

and

$$\frac{s+b}{s+a} = \frac{1}{2a}\left[(a+b) - (b-a)\frac{s-a}{s+a}\right]. \tag{22}$$

Equation (21) is that of a simple lowpass filter, and Eq. (22) describes the circuit of Fig. 1 if a is greater than b. The lefthand side of these equations is the commonly encountered system transfer function $H(s)$, consisting of a frequency-dependent amplitude and phase function. The system transfer function is the frequency transform of the time response to an impulse of voltage $h(t)$; thus,

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt. \tag{23}$$

Normally we think of the system transfer function as the quotient of output to input signal and use this concept to generate the common form expressed by the lefthand side by using a sinewave signal. This concept, however, is only valid if a sinewave is used, since there must in general be a time delay in a network; since Eq. (23) does not contain an explicit time dependence it is apparent that this time discrepancy is absorbed in the complex frequency spectrum and thus locked up so that we cannot readily predict time behavior without mathematical manipulation. The righthand sides of Eqs. (21) and (22) show alternate forms of the system transfer function, obtained purely from a special class of transfer functions which represents a known frequency-dependent time delay without a frequency-dependent amplitude. (Using this form allowed us to unlock the time behavior.)

Examining Eq. (21) it is apparent that the simple lowpass filter can be considered to consist of two parallel constant-amplitude delay functions, one with no delay and the other the delay of Eq. (20). At very high frequencies these two delay signals cancel each other since they arrive at the same time and are of opposing polarity, while at low frequencies there is not a simultaneous output and hence no complete cancellation. A similar interpretation can be placed on Eq. (22).

Thus, the search for a meaningful concept of time delay in a circuit has revealed that there are simple allpass functions which possess a frequency-dependent time delay that fits out intuitive concept of delay; furthermore, a simple minimum-phase network for which the concept of group delay is invalid is now seen to be represented as a combination of allpass delay functions. Figure 3 shows the minimum phase time delay and group

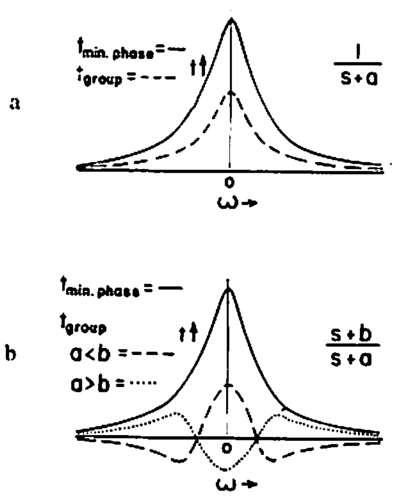


Fig. 3. Minimum phase delay and group delay. The actual minimum phase delay is double-valued and composed of a straight-line zero delay and a bell-shaped delay of the form of Fig. 2. The group delay is single-valued. **a.** Single-pole lowpass circuit. **b.** Single-zero single-pole transfer function.

delay for the single-pole functions of Eqs. (21) and (22). The minimum phase delay is seen to be double-valued for these single-pole functions. The strength of these delayed signals is obtained from the coefficients of Eqs. (21) and (22). It is immediately apparent that group delay is quite misleading for the function of Fig. 1, since this goes to negative time over a substantial portion of the frequency spectrum. The actual delay, as can be seen, never goes negative. Similarly, the group delay of Fig. 3a, although never negative, is nonetheless improper.

A NETWORK CONCEPT

The single-pole single-zero allpass lattice function of Eq. (19) is a primitive function which can be used as a building block for more complicated delays. Two lattices in cascade may, like relations (21) and (22), be composed of combinations of the constituent lattices; for example, if $a \neq b$,

$$\frac{s-a}{s+a}\cdot\frac{s-b}{s+b} = 1 + \frac{a+b}{a-b}\cdot\frac{s-a}{s+a} - \frac{a+b}{a-b}\cdot\frac{s-b}{s+b} \qquad (24)$$

Similarly, one can expand other products of lattices as linear combinations of the individual lattices.

At first glance this would appear to invalidate the conclusion that the time delay of any allpass network is the frequency derivative of the phase function, as the latter is single-valued whereas Eq. (24) shows an expansion which is definitely multiple-valued. Reconciliation may be obtained by remembering that the principle of stationary phase yields the time at which the largest contribution will occur for the integral in Eq. (1). This time will be that of Eq. (7). We might expect that there will be prior contributions and these are discerned in the expansion on the right hand side of Eq. (24). If a sufficiently complicated network of such allpass functions were generated and an oscilloscope used to view the network output with a sudden input transient, the output waveform would be observed to have forerunners preceding the main signal transition. The only condition under which no forerunners would be observed is when the individual lattice sections are identical, in which case there can be no expansion such as Eq. (24). In other words, there is no linear combination for an iterated lattice,

$$[(s-a)/(s+a)]^N \qquad (25)$$

and in this case the delay of Eq. (7) is the only delay. In this special case, if the frequency parameter a is very high, approaching infinity as rapidly as the number of identical sections n, then in the limit as n becomes large without limit this relation becomes the transfer function of ideal delay, [10]

$$e^{-T_0 s} = e^{-iT_0\omega} \text{ for } \sigma = 0. \qquad (26)$$

For all other iterated lattices the delay distribution will be a summation of the constituent delays and in the limit for such a dispersive network will be an integral expression (derived in an earlier paper [1]). The magnitude of terms on the righthand side of Eq. (24) and any such expansion is such that no single term contributes appreciably to the resultant output prior to the time indicated by Eq. (7). Instead each term is effectively nullified by a term representing a prior or later delay, and nullification is not substantially removed until the time of Eq. (7).

From the preceding discussion of forerunners it is quite easy to see how it is possible for a network with the transfer function and time delay of Fig. 3b to be cascaded with a complementing network to produce a constant-gain zero-delay output; thus,

$$(s+b)/(s+a)\cdot(s+a)/(s+b) = 1. \qquad (27)$$

While there is a finite delay component in Eq. (22), there is no necessity to envision a negative time delay to cancel the term of the form (24) which occurs in the cascaded combination, since each and every forerunner except a unity-gain zero-delay forerunner is cancelled completely. Some remarkable facts may now be deduced from the preceding observations about network transfer functions which have all poles and zeros on the real σ axis.

1. Any network with simple poles and zeros restricted to the real σ axis may be considered as equivalent to a *parallel* combination of first-order allpass lattices. There will be one allpass lattice for each pole of the network transfer function. The pole, and hence time delay distribution, of each lattice will be determined by the asso-

ciate transfer function pole, while the strength and polarity of each lattice will be determined by the joint distribution of zeros and poles.

2. Higher order poles in the transfer function will yield series combinations of the associate lattices, with the number of lattice sections determined by the order of the pole.

3. Series combinations of networks may be considered as parallel combinations of the constituent lattices of each network.

Because of the associative property of the Fourier transform, the foregoing conclusions concerning the distribution of equivalent networks mean that since each lattice has a frequency-dependent time delay, the time delay of the network output is not single-valued but a multiple-valued combination of the primitive delays. Fig. 2 is a time-delay frequency distribution for the simple one-pole function. Any other minimum phase network which can be expressed as a rational function factorable to the form

$$\frac{(s+a)(s+b)\ldots}{(s+\alpha)(s+\beta)\ldots} \tag{28}$$

will have a time delay frequency function expressible as a sum of delays of the form of Eq. (20) and will have a graphical plot of delay *vs* frequency such as Fig. 4. To

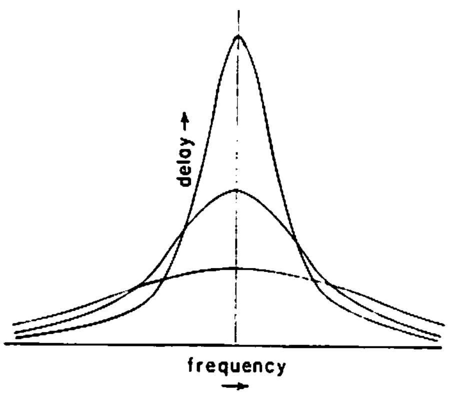


Fig. 4. The multivalued delays to be anticipated for a transfer function with a multiplicity of simple poles at the same frequency.

consider the time delay behavior of such a network, we may thus draw the equivalent network of Fig. 5, where each lattice is considered a frequency-dependent delay line with the delay of Eq. (20). A zero delay may be assumed due to a lattice with a pole at zero frequency. The gain and polarity of each delay line channel is assumed to be determined by a summing amplifier, for the sake of illustration only.

When dealing with a physical process which involves propagation with a frequency-independent velocity, such as sound in air, an equivalent interpretation of Fig. 5 would be that there is a distribution of otherwise perfect sources which assume a frequency-dependent position in space such that the delay due to the additional distance travelled at the velocity of propagation is identical to that of the equivalent delay line.

The allpass lattice of Eq. (19) has a single-pole and single-zero configuration on the real axis. This, as was seen, is quite satisfactory for discussing the time delay of

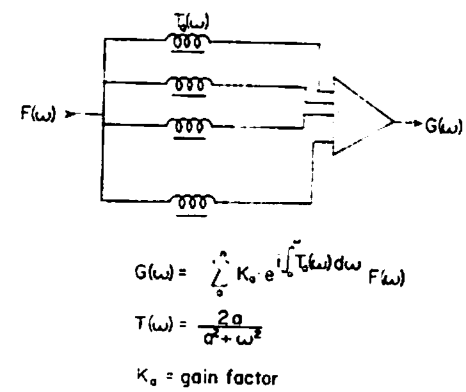


Fig. 5. Symbolic representation of a network with a transfer function expressible as a rational product of terms with poles and zeros. This network may be interpreted as a parallel combination of delay lines with constant amplitude transfer function but a frequency-dependent delay as shown. An input signal with spectral distribution $F(\omega)$ will produce the output $G(\omega)$.

any minimum-phase network with poles on the real axis, i.e., with the terms of Eq. (28) which do not have an imaginary component. A loudspeaker, however, generally has poles with an imaginary component, which leads to peaks and dips in the frequency response and damped ringing in the time response. For this case there exists one type of allpass lattice which, like Eq. (19) on the real axis, can be used to represent the time delay of any network with imaginary poles. This is the second-order lattice with conjugate complex poles and zeros and with the transfer function

$$\frac{(s-a+ib)}{(s+a+ib)}\cdot\frac{(s-a-ib)}{(s+a-ib)}. \tag{29}$$

There does not exist a simple one-parameter delay such as represented by Eq. (20); instead the delay relation now depends upon the position of a and b. The form of delay may be ascertained by allowing the expansion of Eq. (29) to be considered as two cascaded sections of the type of Eq. (19) with appropriate shift in complex frequency. Since the transfer function is now a sum of phase shifts, the time delay from Eq. (20), is [7]

$$t=\frac{2a}{a^2+(\omega-b)^2}+\frac{2a}{a^2+(\omega+b)^2}. \tag{30}$$

Obviously, if the term b approaches zero this becomes the transfer function of Eq. (25) with $n = 2$, so that the delay becomes twice that of Eq. (20). On the other hand, if for a given value of b the term a approaches zero, the phase shift in the vicinity of the frequency of b

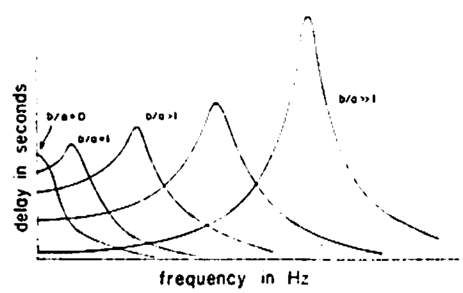
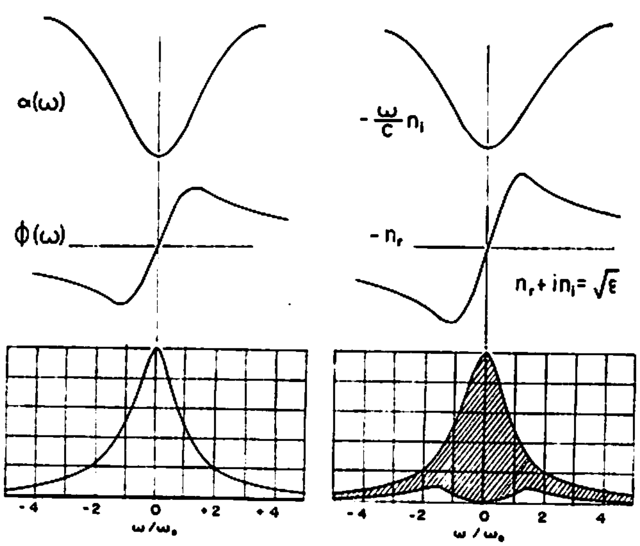


Fig. 6. Representation of the form of the excess delay of a second order allpass lattice.

becomes very large for a small change in frequency. In this case the time delay becomes large without limit. The nature of the delay time for various positions of the poles is shown in Fig. 6.

The form of the delay for the case where b is very much greater than a is the same as in Fig. 2, with the contribution of excess delay occurring at the frequency of b. This leads to the considerable simplification that as long as one is considering local variations in a loudspeaker response, one may consider all activity centered at the frequency of this variation and use the simple expression of Eq. (20). Because local loudspeaker fluctuations in phase and amplitude are usually significant, the equivalent delay and consequently the effective acoustic position relocation may be significant for the frequency of strong local fluctuation. A physical interpretation of this may be secured by observing what would happen if the loudspeaker were fed a transient signal which had in its spectrum this frequency of unusual delay. The pressure wave output would have all frequencies except this component, since for a short time this component will not have arrived. It is a calculable fact that removal of a component is tantamount to adding a cancelling-out of the phase-equivalent component to the original signal. Consequently, the output pressure transient will be perceived to have a "ringing" component at the frequency that is removed. Within some period of time the component frequency will arrive, gracefully one might add, since it is really a distribution of the form of Fig. 2, and the interpretation is that the ringing has now subsided. If the signal is removed from the loudspeaker terminals, the delayed component must persist for some time and the interpretation of this waveform would be that there is a ringing of the output with polarity reversed from the start-up transient.



a b

Fig. 7. **a.** Complete plot of amplitude, phase, and time delay (double valued) for the circuit of Fig. 1 with the frequency of maximum absorption at dc. **b.** Equivalent amplitude and phase characteristic of the transfer function of an electromagnetic wave passing through a single resonance dielectric medium exhibiting anomalous dispersion in which the group velocity by calculation can exceed the velocity of light in vacuum. The frequency dependent time delay (after Brillouin [9]) which has been normalized to the same center frequency of Fig. 7a is a continuum within the shaded region.

ANOMALOUS DISPERSION

A particularly significant distribution of amplitude and phase when discussing group delay is afforded by the transfer function for a real passive dielectric medium with a single simple resonance. The group velocity of a wave propagating in this medium could exceed the velocity of light and gave rise to the term "anomalous dispersion". The concept of group velocity established by Lord Rayleigh was so firmly entrenched that this solution posed a serious challenge to the theory of relativity. So great was this discrepancy that an exceedingly complicated solution was worked by Sommerfeld and Brillouin. Figure 7b is a plot of amplitude, phase, and time delay as worked out by Brillouin [9]. He observed that there was no unique delay, but depending upon sensitivity of apparatus there was a distribution of delays in the shaded region. For comparison with the solution above, Fig. 7a is a similar display for the function of Eq. (22) when a is greater than b. The agreement is quite satisfactory when one realizes that the index of refraction which plays the role of the network transfer function involves a square root of a function of the form of Eq. (22) and hence does not have a simple pole and zero but branch points. The branch points lead to the continuous distribution, whereas simple poles and zeros yield singular functions for time delay.

SUMMARY

A loudspeaker, when considered as a transducer of electrical signals to acoustic pressure, has a transfer function which has a frequency-dependent amplitude and phase response. The effect of these amplitude and phase variations may be considered to be the introduction of a time delay distortion in the reproduced pressure response. The response of an actual loudspeaker will be identical to the response one would have from an ensemble of perfect loudspeakers each one of which assumes a frequency-dependent position in space behind the actual loudspeaker. The number of equivalent loudspeakers, and hence the measure of time delay smearing, will increase with the complexity of the amplitude and phase spectrum. In those portions of the frequency spectrum where the actual loudspeaker is of minimum phase type, it is always possible to modify the response by mechanical or electrical means such that all equivalent loudspeakers merge into one position in space. When this is done there is no frequency-dependent time delay distortion, and the pressure response may be made essentially perfect. Attempts at minimum phase equalization of those portions of the frequency spectrum where the actual loudspeaker is non-minimum phase will not coalesce the equivalent loudspeakers but will leave a spatial distribution which is equivalent to a single perfect loudspeaker with a frequency-dependent position behind the actual loudspeaker.

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